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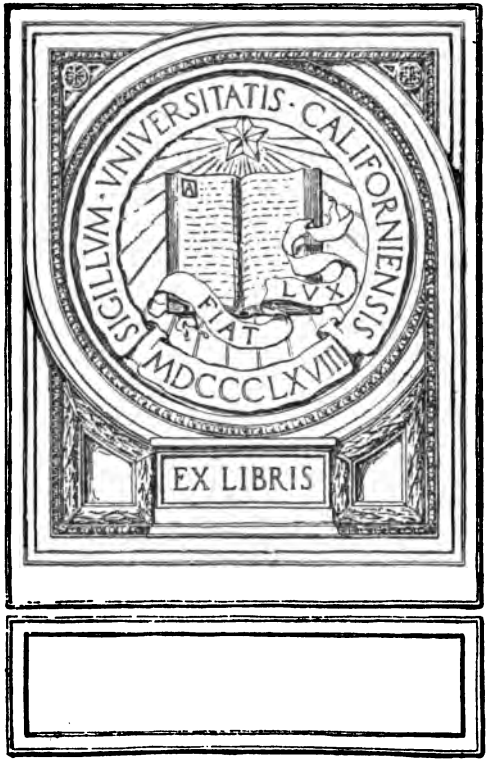

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MANUALS OF TELEGRAPH AND TELEPHONE ENGINEERING

Edited by Sir WILLIAM SLINGO
Late Engineer-in-Chief to General Post Office

TELEPHONIC TRANSMISSION

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Edited by Sir WILLIAM SLINGO

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TELEPHONIC TRANSMISSION

THEORETICAL AND APPLIED

BY

J. G. HILL

ASSISTANT STAFF ENGINEER, GENERAL POST OFFICE, LONDON

WITH 196 DIAGRAMS

LONGMANS, GREEN AND CO.
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EDITOR'S NOTE

ONE of the remarkable features in the literature of the electro-technical world is the paucity of works essaying to deal comprehensively and as completely as our knowledge will permit, with the problems involved in the applications of electricity in the Telegraph and Telephone Services. There are, of course, numerous books of a more or less elementary and general character, and there are several which deal effectively with small fragments of the subject, but no serious attempt has hitherto been made to cover the whole field. That field is obviously too extensive for any attempt to be made to deal with it between the covers of a single volume, and it is specialized to such an extent that no one individual can claim to have an expert knowledge of it in its entirety.

It has therefore been deemed expedient to produce a series of handbooks, so co-ordinated that they shall collectively cover the whole of the ground involved, but that individually they shall be as complete and self-contained as it is possible to make them. It has also been arranged that in order to secure the latest and best available expert information upon the several branches of the subject, the preparation of each handbook shall be entrusted to a specialist. The list of titles of the volumes at present in hand, given on the page facing the title page of this volume, will afford some indication of the scope of the series and of the subject-matter to be dealt with in each volume. The majority of the authors are officers on the Staff of the Engineering Department of the Post Office, and the Postmaster General has been pleased to express his approval of the work which they have undertaken. I have, however, been asked to say that any opinions which the authors may express are personal, and not necessarily those of the Postmaster General or of the Administration.

The theory of telephone transmission is an important subject, and its importance increases with the development of the telephone system. It is natural, therefore, that considerable attention should

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have been devoted to it both at home and abroad, which has resulted in the production of illuminating works by some of its best exponents, but it is believed that no one has hitherto published a book which deals with combined theory and practice, or which provides a bridge between mathematical theory and its application to the everyday problems of the engineer. This is the object of the present volume and the importance and wide application of this method of treatment are considerable. If we accept as a definition of a good engineer, one who can produce a maximum of efficiency at a minimum of cost, then it may be claimed without hesitation, that the book affords considerable assistance in the development of a good telephone engineer. A perusal of its pages will also disabuse the mind of the idea which is prevalent in some quarters, that telephony and other branches of "light" electrical engineering are simple, and free from such abstruse problems as are involved in "heavy" electrical engineering.

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AUTHOR'S PREFACE

THEORETICAL Telephonic Transmission has attracted the attention of competent writers in many countries in recent years. The scientific foundations of the subject were, however, laid with remarkable foresight and accuracy in this country by Oliver Heaviside, more than thirty years ago, and about the same time the subject was independently investigated by Vaschy in France. It was not, however, until the end of the nineteenth century that the first important application of the theory, to the improvement of transmission by series lumped loading, was worked out by Pupin. Since that time the theoretical literature of the subject has grown in volume very materially, both in the form of articles on important branches of the subject, and general treatises covering the whole theory. The author's acknowledgments of his indebtedness to many of the writers in question will be found duly recorded in this volume, but there are many other writers, to whom it was not necessary to specially refer, who have also done valuable work. So far as practical applications of the subject are concerned, the important work done by the American Telephone and Telegraph Co., and the Western Electric Co. of America, are of the highest value, and much of the pioneer work in the important applications of the art has been done in their laboratories. The late National Telephone Co. also did excellent work in this country; and the British Post Office has a trained technical staff, including the late National Telephone Co.'s experts, continually employed on telephonic problems. The Telegraphen-Versuchsamt of the German Post Office has existed for many years, and has published much valuable information from time to time, from its records. The Japanese Post Office has also recently published details of important transmission investigations made in its laboratory. It may further be mentioned that the French Post Office has recently organised a "Service d'Etudes et de Recherches Techniques," and many smaller organisations closely study the most up-to-date methods.

The author takes the opportunity of thanking the Engineer-in-Chief of the Post Office for permission to publish details of Departmental apparatus, and particulars of tests which were carried out by the Post Office Engineering Research Staff.

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The Author is much indebted to Mr. W. J. Hilyer, B.Sc. (Eng.), and to Mr. C. Robinson, B.A. (Eng.), the former having been kind enough to read Chapters XI. and XII., and the latter Chapter XIV.

To Mr. A. B. Morice, B.Sc. (Eng.), the Author's special thanks are due for having worked a large part of the problems, and prepared a number of the curves, besides reading the whole of the proofs, and offering suggestions which must result in making some parts of the book clearer and more complete than they would otherwise have been.

November, 1919.

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INTRODUCTION

IT seems desirable at the outset to describe briefly the special features of this book. These may be summarized as follows :—

(*a*) Numerous examples involving the use of transmission formulæ are given, and fully worked out, and a chapter on transmission measurements is included.

(*b*) A large amount of space is devoted to practical applications of the laws of telephonic transmission, including the solution of important cost problems and standard cable practice.

(*c*) In addition to the usual proofs of transmission equations a special point is made of developing the fundamental formulæ of transmission in what appears to the author to be the simplest way possible, consistent with sound theory.

As regards the last paragraph, the usual method of proof involves a knowledge of sine wave transmission, including the treatment of reflection at the boundary of unequal impedances, and also the differential calculus. This method has doubtless great advantages for those who are sufficiently conversant with it, and accordingly the usual proofs on these lines will be found in Appendix I. and Chapter VII.

There are, however, many persons employed in telephonic work who cannot follow such advanced methods, but to whom nevertheless a clear conception of the fundamental facts underlying the ordinary transmission formulæ would be of great advantage. An attempt has been made to meet this difficulty in some measure. It should, however, be borne in mind, as explained in (*c*), that the simplifications are generally speaking additional matter.

The simplified plan adopted is in the first instance to consider the simplest possible transmission line, *i.e.* the infinite line, and to prove the fundamental formulæ relating to it without the calculus. This is done in Appendix II. Following on from this it is proved, by means of quadratic equations in Appendixes III. and IV., that the constants of an artificial circuit can be found, which is equivalent, within limits which are defined, to a real transmission circuit of any length. In Chapter III. it is shown how the constants of this artificial circuit can be utilized in a simple manner, by methods coming under the operation of Ohm's law, to the building up of the same transmission equations as are developed in Appendix I. and Chapter VII. by the more orthodox methods.

In addition to the development of transmission formulæ by the aid of the laws of reflection and refraction at the junction of unequal

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impedances, transmission losses and gains at such junctions are investigated by expressions for the power in the circuit, and the conditions in which maximum power is obtained are found (see Chapter VII.). This leads to a clear appreciation of what constitutes loss and gain in a telephone circuit. Some unavoidable complexity is here involved, but this is less than that involved in the usual treatment.

It must be recognized from the commencement that there is a certain amount of unavoidable complexity in transmission problems which cannot be eliminated. This may be divided into two classes—

(1) That due to the inherent complexity of the laws of transmission, however developed; and

(2) That due to the fact that telephonic transmission involves alternating current calculations of a complicated kind.

Much of the complexity of the subject is due to the latter cause, and for this reason direct current transmission is first dealt with. There is, however, nothing new in this idea.

Although the use of hyperbolic functions is on the increase in works on telephonic transmission, their use is not universal, and the question may arise as to why they should be extensively used in a book such as this which aims at avoiding unnecessary complexity? The answer is that in the author's opinion their use materially simplifies the working of transmission problems where tables of hyperbolic functions are available. Thus, to take a simple example. The sending end impedance of a transmission line with the distant end closed is $Z_0 \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} = Z_0 \tanh \theta$. If the value of θ is known the value of $\tanh \theta$ may be found direct from a suitable table. The same result may be arrived at by the aid of a table of exponentials such as that given in Appendix VIII., but in that case further arithmetical operations are necessary before the required value is obtained. The fact is that in any finite transmission line, with or without apparatus, every expression for the volts, current, and impedance, at any point of the line, contains a factor or term which can be expressed as a hyperbolic function, and it cannot be expressed in any other way which involves so little labour in evaluation when suitable tables of hyperbolic functions are available, which is assumed. Even, however, if they are not, no *additional* work is caused by the use of the functions. The material advantage pointed out will be found, on studying the subject, to be supplemented by others of an important kind, which are explained in Chapter III.

As regards the importance of telephonic transmission to large telephone administrations, it is only necessary to point out that the subject deals with the efficiency of all the telephones and lines in the system. The plant may cost many millions of pounds, and the most efficient and economical method of laying it out can only be found by the fullest knowledge of all the factors involved. The proper study and application of telephonic transmission must result, therefore, in great economies, as compared with any non-scientific method of providing the plant for a given transmission efficiency.

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CHAPTER I

MATHEMATICAL FORMULÆ AND NOTES

1. Introductory Remarks.—The following formulæ and notes are given with the object of directing the reader's attention to the more important formulæ used in the book, and providing some assistance in their elucidation. The chapter forms a summary of essentials which is not intended to supersede the detailed study of mathematical treatises. If, however, the reader is acquainted with algebra up to and including the theory of indices, the compound interest law, and simple logarithms, and is familiar with ordinary trigonometrical formulæ, he will probably be able to make good use of the notes.

In some cases, however, the differential calculus is used, but as a rule throughout the book, an alternative, and simpler, treatment is indicated with the object of making the fundamental parts of the subject accessible to as large a number of readers as possible.

2. The Binomial Theorem.—This theorem provides a formula which makes it possible to immediately raise any binominal quantity such as $(a+b)$ to any proposed power. The theorem stated algebraically is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1.2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3 + \dots \quad (1)$$

The number of terms on the right-hand side of this equation is $n+1$ when n is positive, the number of terms is infinite when n is negative or fractional.

The accuracy of the equation may easily be verified by using it to develop well known formulæ, such, for example, as $(a+b)^3$ or any more complex power of $(a+b)$.

3. The Exponential Series.—It may be seen from paragraph 2 that if n is greater than 1, no matter how large the index n may be, the number of terms is $n+1$, so that if n be infinitely great, the number of terms will be infinite, and we thus obtain an infinite series. If such a series tends to a definite limiting value, as the number of terms is indefinitely increased, the series is said to be convergent. Inasmuch as the supposed series contains an infinite number of terms, it is obvious that perfect exactness in calculation cannot be attained, but by taking a sufficient number of terms any desired degree of exactitude may be arrived at. Now the terms of some series decrease much more rapidly in value than others, and the one we are about to deal with is a rapidly converging series. Equation (1) may be made use of. As a first step in its

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development $(a+b)^n$ is modified as follows. Let $a=1$, $b=\frac{1}{x}$ and $n=x$, and let the sum of the series be called e, then we have

$$(a+b)^n=\left(1+\frac{1}{x}\right)^x=e$$

If x is a very large quantity and if this algebraical series be expanded by the binomial theorem, we obtain

$$e=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\frac{1}{1.2.3.4.5}+\ldots \quad (2)$$

This expansion converges so rapidly that if the sum of the first 13 terms be taken, the value of e is correctly found to seven decimal places, or $e=2{\cdot}7182818$.

In any equation of the form $y=a^x$, where a is any constant and x any variable exponent, y is called an exponential expression. If the constant has the value e already developed, we have $y=e^x$. Now e is the base of the Napierian logarithms, also known as the natural, and the hyperbolic logarithms. If e be raised to the power x by the aid of (2) and the binomial theorem, we have

$$e^x=\left(1+\frac{1}{x}\right)^{x^2}=1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\frac{x^5}{1.2.3.4.5}+\ldots \quad (3)$$

This is the exponential series, and it will be used in connection with the sine and cosine series.

Napierian logarithms, and the expression e^x, are very largely used in analytical investigations on account of some remarkable properties which they possess, and which make for simplification. A remarkable property of equation (3), for example, which is of use in telephonic transmission is at once made evident by a simple application of the differential calculus. By differentiation of (3) we obtain

$$\frac{d(e^x)}{x}=0+\frac{1}{1}+\frac{2x}{1.2}+\frac{3x^2}{1.2.3}+\frac{4x^3}{1.2.3.4}+\frac{5x^4}{1.2.3.4.5}+\ldots$$

So that the series is unchanged by differentiation; or in other words, $\frac{d(e^x)}{x}=e^x$, that is to say, the rate of change of e^x with respect to x is e^x. If any other base than e be taken, the rate of change is more complex.

An application of an exponential equation to telephonic transmission is given at the end of the chapter.

4. Relations between the Sine, Cosine, and Exponential.—The sine and cosine of an angle x, which are functions of the angle in circular trigonometry, may be expressed as expansions in series, thus—

$$\sin x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\ldots \quad (4)$$

$$\cos x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\ldots \quad (5)$$

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If jx be substituted for x in formula (3) we obtain

$$e^{jx} = 1 + jx - \frac{x^2}{1.2} - \frac{jx^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \frac{jx^5}{1.2.3.4.5} - \ldots .$$

$$= \left(1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \ldots\right) + j\left(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots\right) \quad (6)$$

that is

$$e^{jx} = \cos x + j \sin x \quad (7)$$

(7) is known as De Moivre's theorem.
It may be similarly shown that

$$\frac{1}{e^{jx}} = e^{-jx} = \cos x - j \sin x \quad (8)$$

5. Geometrical Progression.—In the following equation let S be the sum of n terms in series

$$S = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \quad (9)$$

Here any term is always r times the preceding term, and r is called the common ratio. Since the second term is ar, the nth term is ar^{n-1}. Multiply each side of (9) by r and we obtain

$$Sr = ar + ar^2 + ar^3 + \ldots + ar^n \quad (10)$$

Subtract (9) from (10) and we obtain

$$Sr - S = ar^n - a \quad (11)$$

seeing that the other terms cancel out. If r is greater than 1 we have

$$S(r - 1) = ar^n - a$$

whence

$$S = \frac{a(r^n - 1)}{r - 1} \quad (12)$$

If r is less than 1 this becomes

$$S - Sr = a - ar^n$$

whence

$$S = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r} \quad (13)$$

It is clear from (13) that S, the sum of the geometrical series, is less than $\frac{a}{1-r}$, but if n is made sufficiently large $\frac{ar^n}{1-r}$ will become very small, and in this way S may be made to differ from $\frac{a}{1-r}$ by a quantity which is less than any assignable quantity, therefore in the limit

$$S = \frac{a}{1 - r} \quad (14)$$

6. Vectors—General.—Physical quantities that have magnitude, direction, and sense are called vector quantities, or simply vectors.

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Thus, alternating voltages and currents continuously vary in magnitude and phase (corresponding to direction), and are alternately positive and negative (corresponding to sense), and may therefore be represented by vectors. The vector OP (Fig. 1) has a magnitude OP, a direction which is indicated by the angle ϕ that OP makes with the horizontal line a and a sense indicated by the arrow head in OP.

A vector of a given magnitude may have any direction, and it is convenient for our purpose to regard its magnitude as the hypotenuse of a right-angled triangle, as shown in Fig. 1, where OP is opposite the right angle. If O be regarded as the centre of a circle round which OP turns, the component parts a and b of the vector will be different with every position; but in all cases, except when the line OP is vertical or horizontal, it is possible to construct a right-angled triangle with OP as hypotenuse, and with the side a horizontal and the side b vertical. In such a case the horizontal line serves as a fixed axis of reference, and this facilitates calculation, and comparison of one vector with another. If the magnitude of OP be called A, then this magnitude may be represented in different ways as follows :—

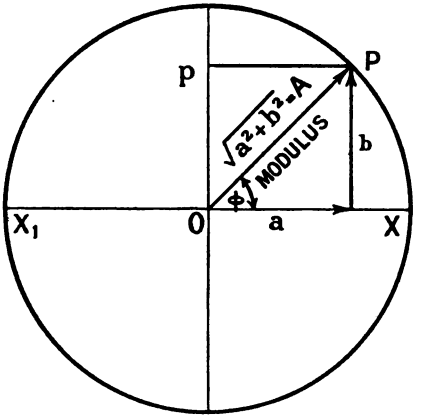

FIG. 1.

(*a*) A may be regarded by Euclid I. 47 as having a value $\sqrt{a^2+b^2}$, and this magnitude is at an angle ϕ with OX. If the angle is positive the vector value may then be written

$$A/\underline{\phi} = \sqrt{a^2+b^2}/\underline{\tan^{-1}\frac{b}{a}} \quad . \quad . \quad . \quad . \quad (15)$$

that is, $\sqrt{a^2+b^2}$ at a positive angle ϕ whose tangent is $\frac{b}{a}$. A negative angle is indicated thus $\overline{\phi}$.

(*b*) If in Fig. 1, O is the centre of a circle of which the diameter is x_1x, all lines from o to x are regarded as positive and lines from o to x_1 as negative. A line turned through a right angle in the positive direction, such as b, is denoted by $\sqrt{-1}=j$ to indicate the degree of turning, hence the vertical line b in Fig. 1 may be denoted by jb. If the rotation had been 90° in the negative direction the vector would have been $-jb$. The vector $A/\underline{\phi}$ may be regarded as the resultant of $a+jb$, and we then have

$$A/\underline{\phi} = a+jb \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Powers of $\sqrt{-1}$ . . . . . (17)

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Since $j = \sqrt{-1}$ $\therefore j^2 = -1$, $j^3 = -\sqrt{-1}$, $j^4 = +1$, etc.
From Fig. 1 we have

$$a = A\cos\phi;\ b = A\sin\phi;\ \text{and}\ \therefore jb = Aj\sin\phi$$

Adding the value of a and jb, and collecting the results of (7), (15) and (16), we have

$$a + jb = A(\cos\phi + j\sin\phi) = Ae^{j\phi} = \sqrt{a^2+b^2}\,\underline{/\tan^{-1}\frac{b}{a}} = A\underline{/\phi} \quad (18)$$

In this case the angle ϕ has been substituted for x in (7), and this course will be adhered to in future references.

Subtracting jb from a we may similarly write

$$a - jb = A(\cos\phi - j\sin\phi) = Ae^{-j\phi} = \sqrt{a^2+b^2}\,\overline{\tan^{-1}\left(-\frac{b}{a}\right)} = A\overline{\phi} \quad (19)$$

In Fig. 1 $A = \sqrt{a^2+b^2}$ is known as the *modulus*, and

$$\phi = \tan^{-1}\frac{b}{a} = \cos^{-1}\frac{a}{A} = \sin^{-1}\frac{b}{A} \text{ is the } \textit{argument} \quad (20)$$

Operations on Vectors.

7. Addition and Subtraction of Vectors.—$a + jb$ is a complex number having a real part a and an imaginary part jb. If two complex numbers are equal, their real parts are equal and their imaginary parts are equal, thus: if $a + jb = c + jd$, then $a = c$ and $b = d$.

If $A = (a + jb) + (c + jd)$, then

$$A = (a + c) + j(b + d) \quad . \quad . \quad . \quad . \quad (21)$$

To add vectors, therefore, add all the real parts to obtain the real part of the resultant, and add all the imaginary parts to obtain the imaginary part of the resultant.

The same principle applies to the **subtraction of vectors**: thus If $A = (a + jb) - (c + jd)$, then

$$A = (a - c) + j(b - d) \quad . \quad . \quad . \quad . \quad (22)$$

8. Multiplication of Vectors.—By (18) we have $a + jb = Ae^{j\phi}$.

Similarly we may write $c + jd = Be^{j\phi_1}$.

$$\therefore (a + jb)(c + jd) = Ae^{j\phi} \times Be^{j\phi_1} = ABe^{j(\phi+\phi_1)} = AB\underline{/\phi + \phi_1} \quad (23)$$

Hence the product of vectors is found by multiplying their magnitudes or moduli and adding their angles.

9. The Quotient of two Vectors is similarly found by dividing one modulus by the other, and subtracting the angle or argument of the divisor from that of the dividend; thus

$$\frac{A}{B}\cdot\frac{(\cos\phi + j\sin\phi)}{(\cos\phi_1 + j\sin\phi_1)} = \frac{A}{B}\{\cos(\phi - \phi_1) + j\sin(\phi - \phi_1)\}$$

or

$$\frac{a + jb}{c + jd} = \frac{Ae^{j\phi}}{Be^{j\phi_1}} = \frac{A}{B}e^{j(\phi - \phi_1)} = \frac{A\underline{/\phi}}{B\underline{/\phi_1}} = \frac{A}{B}\underline{/\phi - \phi_1}. \quad (24)$$

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10. Powers of a Vector.—

Example.—Raise $A(\cos\phi + j\sin\phi) = Ae^{j\phi}$ to the nth power.

We have $$A^n(\cos n\phi + j\sin n\phi) = A^n e^{jn\phi} = (a+jb)^n . \quad (25)$$

This is proved in works on trigonometry. A is always positive, ϕ is any angle, and n any number, positive or negative. The angle ϕ is usually expressed in radians.

11. The Square Root of a Vector.—As a particular case of (25), we have

$$A^{\frac{1}{2}}\left(\cos\frac{\phi}{2} + j\sin\frac{\phi}{2}\right) = \sqrt{A}e^{j\frac{\phi}{2}} = \sqrt{a+jb} = \sqrt{A}\angle\frac{\phi}{2} \quad (26)$$

Hence, to obtain the square root of a vector take the square root of the modulus and divide the angle by 2.

12. Logarithm of a Vector.—

(*a*) If $\log_e(a+jb) = \log_e\sqrt{a^2+b^2}\,\angle\tan^{-1}\frac{b}{a}$ [see (18)] $= A + jB$

then $$A = \tfrac{1}{2}\log_e(a^2+b^2); \text{ and } B = \tan^{-1}\frac{b}{a} . \quad . \quad (27)$$

(*b*) Let $(a+jb) = A_1 e^{j\phi}$. Since $j\phi$ is $\log_e$ of $e^{j\phi}$.

$$\therefore \log_e(a+jb) = \log A_1 + j\phi$$

where ϕ is expressed in radians.

13. Hyperbolic Functions.—The mathematical expressions which show the relations of voltage and current in finite transmission lines contain functions of the exponential which are known as hyperbolic functions, and these may be derived as follows.

By addition of equations (7) and (8) we have

$$e^{j\phi} + e^{-j\phi} = \cos\phi + j\sin\phi + \cos\phi - j\sin\phi = 2\cos\phi$$

whence $$\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2} . \quad . \quad . \quad . \quad . \quad (28)$$

Similarly by subtraction of (8) from (7) we obtain

$$\sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{2j} . \quad . \quad . \quad . \quad . \quad (29)$$

If $j\theta$ be now written for ϕ in (28) and (29), we obtain

$$\cos\phi = \cos j\theta = \frac{e^{-\theta} + e^{\theta}}{2} . \quad . \quad . \quad . \quad (30)$$

$$\sin\phi = \sin j\theta = \frac{1}{j}\left(\frac{e^{-\theta} - e^{\theta}}{2}\right) = j\left(\frac{e^{\theta} - e^{-\theta}}{2}\right) . \quad (31)$$

The expressions $\frac{e^{\theta} + e^{-\theta}}{2}$ and $\frac{e^{\theta} - e^{-\theta}}{2}$ are called cosh θ and sinh θ respectively. We may therefore write by (30) and (31)

$$\cos j\theta = \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} . \quad . \quad . \quad (32)$$

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$$\sin j\theta = j \sinh \theta = j\left(\frac{e^{\theta} - e^{-\theta}}{2}\right)$$

or
$$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The similarity between the sine and cosine and the sinh and cosh values extends also to the geometrical relations between the two as well as to the algebraic values of these functions. It can, in fact, be shown that there are six hyperbolic functions connected with the equilateral hyperbola, which have a resemblance to the six functions of circular trigonometry, and which are called the hyperbolic sine, cosine, tangent, secant, cosecant, and cotangent. These functions receive the general name of hyperbolic functions, and are as follows; the whole six are given together for reference.

$$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

$$\tanh \theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \quad . \quad . \quad . \quad . \quad . \quad (36)$$

$$\operatorname{cosech} \theta = \frac{2}{e^{\theta} - e^{-\theta}} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

$$\operatorname{sech} \theta = \frac{2}{e^{\theta} + e^{-\theta}} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

$$\coth \theta = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}} \quad . \quad . \quad . \quad . \quad . \quad (39)$$

It should be noticed that in every case the names given to these functions are the same as those given to the corresponding trigonometrical function, with the addition of the letter *h*, which indicates that they are hyperbolic functions.

From the fact that $\cos j\theta = \cosh \theta = \cos \phi$ and $\sin j\theta = j \sinh \theta = \sin \phi$ by (30) and (31), it follows that any formula in circular functions may be transformed into one in hyperbolic functions, but the + and − signs in the two sets of formulæ will not necessarily be the same; cosh θ has, however, the same sign as cos ϕ and replaces it; but sinh θ requires further examination, seeing that $j \sinh \theta = \sin \phi$. So long as these functions are directly equated to each other and involve only the first power, the operator j remains. There are, however, a number of equations in which the first power of j sinh θ appears on each side, and cancels out. For example, we have

$$\frac{\sin \phi}{\cos \phi} = \frac{j \sinh \theta}{\cosh \theta} = j \tanh \theta \text{ or } \frac{\sinh \theta}{\cosh \theta} = \tanh \theta \, . \quad (40)$$

Also
$$\frac{1}{j \sinh \theta} = \frac{\operatorname{cosech} \theta}{j} \text{ or } \frac{1}{\sinh \theta} = \operatorname{cosech} \theta \, . \quad . \quad (41)$$

If the second power of j sinh θ appears in any expression, it

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becomes $(j \sinh \theta)^2 = -\sinh^2 \theta$, and similar treatment may be applied to other functions containing j^2. Bearing these facts in mind, any equation in ordinary trigonometry may be converted into hyperbolic functions; for example—

$$\cos^2 \phi + \sin^2 \phi = 1 \text{ becomes by (30) to (33)}$$
$$\cosh^2 \theta + (j \sinh \theta)^2 = \cosh^2 \theta - \sinh^2 \theta = 1$$

$\tan 2\phi = \dfrac{2 \tan \phi}{1 - \tan^2 \phi}$ may be expressed in hyperbolic functions thus

$j \tanh 2\theta = \dfrac{2j \tanh \theta}{1 - (j \tanh \theta)^2}$; that is, $\tanh 2\theta = \dfrac{2 \tanh \theta}{1 + \tanh^2 \theta}$

Proceeding in this way the following relations of hyperbolic functions to each other may be deduced. [See also (34) to (39).]

14. List of Formulæ of Hyperbolic Functions with Multiple, etc., Angles.—

$$\sinh (\theta \pm \theta_1) = \sinh \theta \cosh \theta_1 \pm \cosh \theta \sinh \theta_1 \quad . \quad . \quad (42)$$

$$\sinh (\theta + \theta_1 + \theta_2) = \sinh \theta_1 \cosh \theta \cosh \theta_2 + \sinh \theta \sinh \theta_1 \sinh \theta_2 + \sinh \theta \cosh \theta_1 \cosh \theta_2 + \sinh \theta_2 \cosh \theta \cosh \theta_1 \quad . \quad . \quad . \quad (43)$$

$$\sinh \theta + \cosh \theta = e^{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$\cosh \theta - \sinh \theta = e^{-\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\cosh^2 \theta - \sinh^2 \theta = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

$$\sinh 2\theta = 2 \sinh \theta \cosh \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$$\cosh (\theta \pm \theta_1) = \cosh \theta \cosh \theta_1 \pm \sinh \theta \sinh \theta_1 \quad . \quad . \quad (48)$$

$$\cosh 2\theta = \cosh^2 \theta + \sinh^2 \theta = 2 \sinh^2 \theta + 1 = 2 \cosh^2 \theta - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\operatorname{sech}^2 \theta = 1 - \tanh^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

$$\operatorname{cosech}^2 \theta = \operatorname{cotanh}^2 \theta - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$\sinh \frac{\theta}{2} = \sqrt{\tfrac{1}{2}(\cosh \theta - 1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$\cosh \frac{\theta}{2} = \sqrt{\tfrac{1}{2}(\cosh \theta + 1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

$$\tanh \frac{\theta}{2} = \frac{\sinh \theta}{1 + \cosh \theta} = \frac{\cosh \theta - 1}{\sinh \theta} \quad . \quad . \quad . \quad . \quad (54)$$

$$\tanh 2\theta = \frac{2 \tanh \theta}{1 + \tanh^2 \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$\tanh (\theta \pm \theta_1) = \frac{\tanh \theta \pm \tanh \theta_1}{1 \pm \tanh \theta \tanh \theta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

The list might, of course, be continued if necessary.

15. Maxima and Minima Values of the Hyperbolic Functions.— As θ changes from $-\infty$ to $+\infty$, all intermediate values between the limits are attained.

$$\sinh 0 = 0 \qquad \cosh 0 = 1 \qquad \tanh 0 = 0 \quad . \quad . \quad (57)$$

$$\sinh \infty = \infty \qquad \cosh \infty = \infty \qquad \tanh \infty = 1 \quad . \quad . \quad (58)$$

$$\sinh (-\infty) = -\infty \qquad \cosh (-\infty) = \infty \qquad \tanh (-\infty) = -1 \quad (59)$$

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16. Differential Coefficients and Integrals of Hyperbolic Functions.

$$\frac{d \sinh \theta}{d\theta} = \cosh \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$\frac{d \cosh \theta}{d\theta} = \sinh \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

$$\frac{d \tanh \theta}{d\theta} = \operatorname{sech}^2 \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$\frac{d \log_e \sinh \theta}{d\theta} = \coth \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

$$\frac{d \log_e \cosh \theta}{d\theta} = \tanh \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$\frac{d \log_e \tanh \theta}{d\theta} = \operatorname{sech} \theta \operatorname{cosech} \theta = 2 \operatorname{cosech} 2\theta \quad . \quad (65)$$

$$\int \sinh^2 \theta d\theta = \tfrac{1}{2}(\sinh \theta \cosh \theta - \theta) + c \quad . \quad . \quad (66)$$

$$\int \cosh^2 \theta d\theta = \tfrac{1}{2}(\sinh \theta \cosh \theta + \theta) + c \quad . \quad . \quad (67)$$

$$\int \tanh^2 \theta d\theta = (\theta - \tanh \theta) + c \quad . \quad . \quad . \quad . \quad . \quad (68)$$

17. Complex Hyperbolic Angles.—In alternating current problems expressions are met with which involve imaginary hyperbolic angles: thus we have $\sinh (\theta + j\theta_1) = \sinh \theta \cosh j\theta_1 + \cosh \theta \sinh j\theta_1$.

To obtain the value of this equation the hyperbolic angles $\cosh j\theta_1$ and $\sinh j\theta_1$ are converted as follows. We have from (32) $\cosh \theta_1 = \cos j\theta_1$. If θ_1 be changed to $j\theta_1$ this becomes

$$\cosh j\theta_1 = \cos j^2\theta_1 = \cos (-\theta_1) = \cos \theta_1 \quad . \quad (69)$$

We also have by (33) $j \sinh \theta_1 = \sin j\theta_1$. If θ_1 be changed to $j\theta_1$ we obtain

$$j \sinh j\theta_1 = \sin (-\theta_1)$$

or
$$\sinh j\theta_1 = -j \sin (-\theta_1) = j \sin \theta_1 \quad . \quad . \quad (70)$$

Substituting the values of $\cosh j\theta_1$ and $\sinh j\theta_1$ in the equation for $\sinh (\theta + j\theta_1)$ from (69) and (70), we obtain

$$\sinh (\theta + j\theta_1) = \sinh \theta \cos \theta_1 + j \cosh \theta \sin \theta_1$$

and similarly

$$\cosh (\theta + j\theta_1) = \cosh \theta \cos \theta_1 + j \sinh \theta \sin \theta_1$$

also
$$\tanh (\theta \pm j\theta_1) = \frac{\tanh \theta \pm \tanh j\theta_1}{1 \pm \tanh \theta \tanh j\theta_1}$$

or
$$\tanh (\theta \pm j\theta_1) = \frac{\tanh \theta \pm j \tan \theta_1}{1 \pm j \tanh \theta \tan \theta_1}$$

Since by (69) and (70) $\dfrac{\sinh j\theta_1}{\cosh j\theta_1} = \dfrac{j \sin \theta_1}{\cos \theta_1}$

or
$$\tanh j\theta_1 = j \tan \theta_1 \quad . \quad . \quad . \quad . \quad (71)$$

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18. List of Formulæ involving Imaginary Hyperbolic Angles.—Collecting and extending the results of the expressions for imaginary hyperbolic angles, we have the following list:—

$$\sinh j\theta_1 = j \sin \theta_1 \qquad \sin j\theta_1 = j \sinh \theta_1 \quad . \quad . \quad . \quad (72)$$

$$\cosh j\theta_1 = \cos \theta_1 \qquad \cos j\theta_1 = \cosh \theta_1 \quad . \quad . \quad . \quad (73)$$

$$\tanh j\theta_1 = j \tan \theta_1 \qquad \tan j\theta_1 = j \tanh \theta_1 \quad . \quad . \quad . \quad (74)$$

$$\coth j\theta_1 = -j \cot \theta_1 \qquad \cot j\theta_1 = -j \coth \theta_1 \quad . \quad . \quad (75)$$

$$\operatorname{sech} j\theta_1 = \sec \theta_1 \qquad \sec j\theta_1 = \operatorname{sech} \theta_1 \quad . \quad . \quad . \quad (76)$$

$$\operatorname{cosech} j\theta_1 = -j \operatorname{cosec} \theta_1 \qquad \operatorname{cosec} j\theta_1 = -j \operatorname{cosech} \theta_1 \quad . \quad (77)$$

$$\sinh (\theta \pm j\theta_1) = \sinh \theta \cos \theta_1 \pm j \cosh \theta \sin \theta_1 \quad . \quad . \quad . \quad (78)$$

$$\cosh (\theta \pm j\theta_1) = \cosh \theta \cos \theta_1 \pm j \sinh \theta \sin \theta_1 \quad . \quad . \quad . \quad (79)$$

$$\tanh (\theta \pm j\theta_1) = \frac{\tanh \theta \pm j \tan \theta_1}{1 \pm j \tanh \theta \tan \theta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

19. Theorem relating to Hyperbolic Functions.—If

$$\frac{b}{a} = \frac{\sinh \theta_1}{\cosh \theta_1}$$

then
$$\sqrt{a^2 - b^2} \sinh (\theta + \theta_1) = a \sinh \theta + b \cosh \theta \quad . \quad (81)$$

From the preceding we have

$$\frac{b^2}{a^2} - 1 = \frac{\sinh^2 \theta_1}{\cosh^2 \theta_1} - 1$$

whence
$$\frac{b^2 - a^2}{a^2} = \frac{\sinh^2 \theta_1 - \cosh^2 \theta_1}{\cosh^2 \theta_1} = -\frac{1}{\cosh^2 \theta_1} \text{ by (46)}$$

and
$$\cosh \theta_1 = \frac{a}{\sqrt{a^2 - b^2}} \quad . \quad . \quad . \quad . \quad (82)$$

Also similarly we may write by inverting $\frac{b}{a}$

$$\frac{a^2 - b^2}{b^2} = \frac{\cosh^2 \theta_1 - \sinh^2 \theta_1}{\sinh^2 \theta_1} = \frac{1}{\sinh^2 \theta_1}$$

whence
$$\sinh \theta_1 = \frac{b}{\sqrt{a^2 - b^2}} \quad . \quad . \quad . \quad . \quad (83)$$

Substitute the values of $\sinh \theta_1$ and $\cosh \theta_1$ from (82) and (83) in (42), and we obtain

$$\sinh (\theta + \theta_1) = \sinh \theta \frac{a}{\sqrt{a^2 - b^2}} + \cosh \theta \frac{b}{\sqrt{a^2 - b^2}}$$

that is
$$\sqrt{a^2 - b^2} \sinh (\theta + \theta_1) = a \sinh \theta + b \cosh \theta \quad . \quad (84)$$

20. Maxima and Minima.—If y is a function of x there are numerous cases where y increases to a maximum value and then decreases, or where y decreases to a minimum and then increases. Such cases are frequently met with in transmission problems, and two methods of finding the maxima and minima values will now be explained. There are also more complex cases where maxima

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and minima values alternate and recur, but these are not dealt with here. The problems dealt with have a direct application at a later stage. Problem 1 is used frequently, and problem 2 is utilised in Chapter IX.

Problem 1.—Find a minimum value for $\frac{1}{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

we have
$$y=\frac{\sqrt{x}+\frac{1}{\sqrt{x}}}{2} \quad . \quad . \quad . \quad . \quad . \quad (85)$$

If a curve be plotted with values of y as ordinates and values of x as abscissæ, the minimum value may be read by inspection of

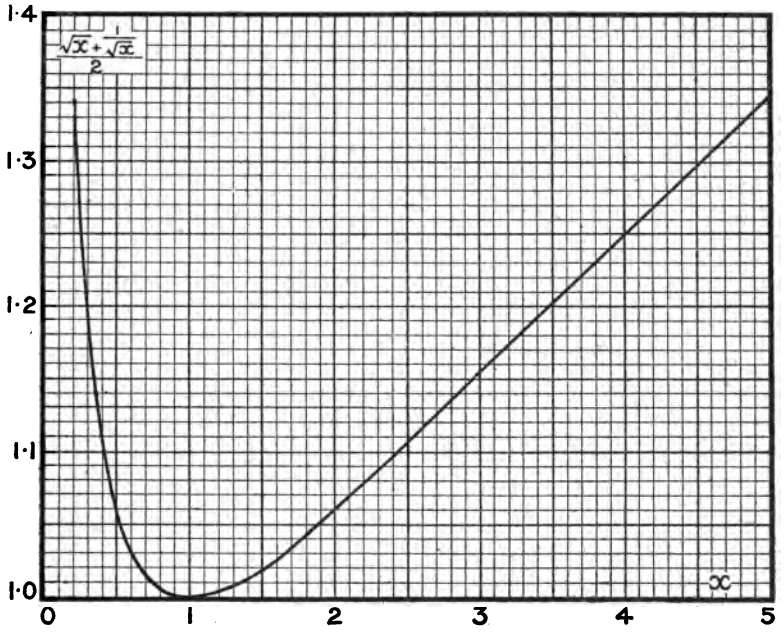

FIG. 2.

the curve. It will suffice in this case to plot values of x between 0 and 5, as it is obvious by inspection that $\sqrt{x}$ continually increases and $\frac{1}{\sqrt{x}}$ continually decreases, and therefore there can only be one minimum value of y; the maximum value of y will only be reached when x reaches infinity on the one hand, and is infinitely small on the other hand. Fig. 2 shows that the minimum value of y is attained when $x=1$. To find the maximum or minimum by differentiation in this case, we have

$$\frac{dy}{dx}=\frac{1}{4}\left(\frac{1}{x^{\frac{1}{2}}}-\frac{1}{x^{\frac{3}{2}}}\right)=\frac{1}{4x}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right) \quad . \quad . \quad (86)$$

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When the minimum value of y is reached it commences to increase, and at the exact value $\frac{dy}{dx} = 0$ (86) becomes

$$0 = \frac{1}{4x}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) \quad \text{or} \quad \sqrt{x} = \frac{1}{\sqrt{x}} \quad . \quad . \quad (87)$$

and therefore $x = 1$.

In order to find whether x has a maximum or minimum value when $\frac{dy}{dx} = 0$ differentiate a second time, so as to obtain the value of $\frac{d^2y}{dx^2}$, and substitute the value of x so found. If the result is negative, y is a maximum; if it is positive, y is a minimum. Applying this rule to (86) we have

$$\frac{d^2y}{dx^2} = \frac{3}{8x^{\frac{5}{2}}} - \frac{1}{8x^{\frac{3}{2}}} = \frac{1}{8x^{\frac{3}{2}}}\left(\frac{3}{x} - 1\right) \quad . \quad . \quad . \quad (88)$$

When x has the critical value 1 this is positive, and the value of y at that point is therefore a minimum.

The equation $\frac{1}{2}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = y$ is one of a type which will repay further examination, as similar equations are used many times in this book. It will now be shown that any equation of the form

$$y = \frac{a}{x^n} + bx^n \quad . \quad . \quad . \quad . \quad . \quad (89)$$

where a and b are constants, x is a variable, and n any constant whatever positive or negative, has a minimum value when $\frac{a}{x^n} = bx^n$, and this can be proved graphically in the same way as Fig. 2. We have by differentiation

$$\frac{dy}{dx} = -anx^{-(n+1)} + bnx^{n-1}$$

For a minimum value we have

$$anx^{-(n+1)} = bnx^{n-1}; \text{ that is } \frac{a}{b} = \frac{x^{n-1}}{x^{-(n+1)}} = x^{2n}$$

or

$$\frac{a}{x^n} = bx^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

If $a = b = \frac{1}{2}$ and $n = \frac{1}{2}$ (89) becomes $y = \frac{1}{2}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ which is the example shown in the graph, and when $x = 1$ it will be seen that the equality given in (90) holds good.

If $\frac{1}{a} = b$ (89) becomes

$$y = \frac{a}{x^n} + \frac{x^n}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

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and we have the case of a fraction $\frac{x^n}{a}$ and its reciprocal. It follows that the minimum value of any such fraction when added to its reciprocal is attained when the sum of the fraction and its reciprocal $= 2$, for by (90) $\frac{a}{x^n} = \frac{x^n}{a}$, whence $a = x^n$, and therefore $\frac{a}{x^n} = 1$ and $\frac{x^n}{a} = 1$. We have, therefore, $\frac{x^n}{a} + \frac{a}{x^n} = 2$ in such a case and $\frac{x^n}{a} - \frac{a}{x^n} = 0$.

If instead of x^n we take a^x, then in the equation

$$y = a^x + \frac{1}{a^x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

where x may have any value positive or negative, and a may be any constant whatever, the minimum value of y is again reached when $a^x = \frac{1}{a^x}$. It will be noticed that we have again the sum of a number and its reciprocal. By plotting values of x against y from (92) the minimum value may be found. By differentiation of (92) we have

$$\frac{dy}{dx} = a^x \log_e a - a^{-x} \log_e a \quad . \quad . \quad . \quad . \quad (93)$$

For a minimum value we have

$$a^x \log_e a = a^{-x} \log_e a$$

whence $a^x = \frac{1}{a^x}$ or $(a^x)^2 = 1$ and therefore $a^x = 1$ and $x = 0$.

It follows that

$$a^x + \frac{1}{a^x} = 2 \text{ and } a^x - \frac{1}{a^x} = 0 \quad . \quad . \quad . \quad (94)$$

If in (92) $a = e$, the root of the natural logarithms, we have

$$y = e^x + e^{-x} = 2 \cosh x$$

and also $$y_1 = e^x - e^{-x} = 2 \sinh x$$

It follows from (94) that the minimum value of $\cosh x$ is 1 and also that the minimum value of $\sinh x = 0$.

Therefore the minimum value of

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{0}{1} = 0 \quad . \quad . \quad . \quad . \quad (95)$$

and this agrees with (57).

Problem 2.—In the expression

$$y = \log_e \frac{D}{d} \times d^2 \quad . \quad . \quad . \quad . \quad . \quad (96)$$

where D is a constant and d a variable less than D, show that the maximum value of y is obtained when $\log_e \frac{D}{d} = 0{\cdot}5 = \log_e \sqrt{e}$. This

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is solved graphically in Fig. 3, where the values of d are plotted as abscissæ and the values of y are given as ordinates, the value of D being taken as 10.

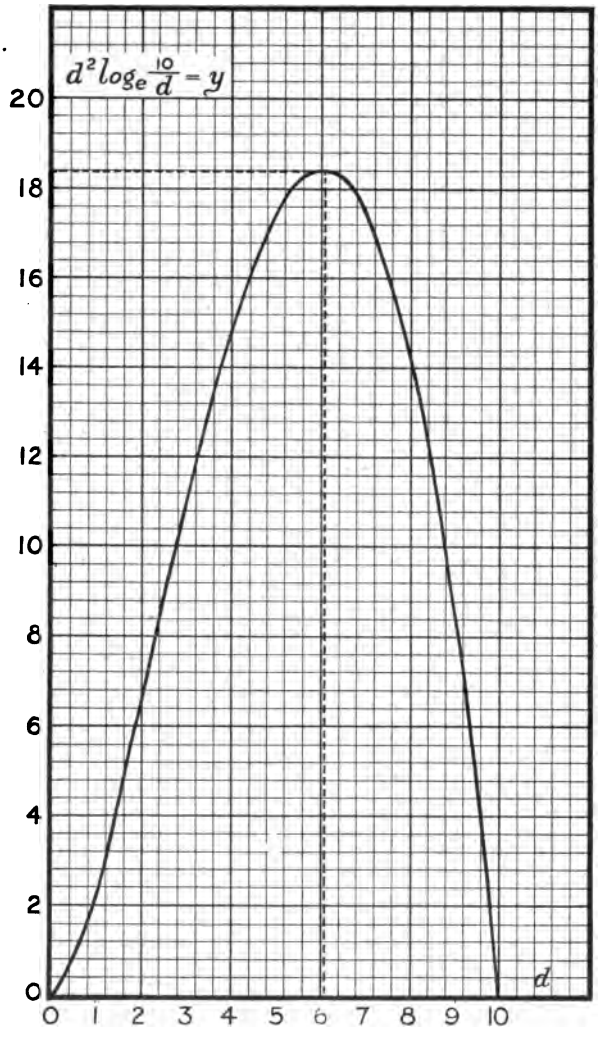

FIG. 3.

To solve by the differential calculus use the well-known formula $d\frac{uv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$, both u and v being functions of x, then if $\log_e \frac{D}{d} = u$ and $d^2 = v$, we have

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$$\frac{dy}{dd} = 2d \log_e \frac{D}{d} - d^2 \cdot \frac{d}{D} \cdot \frac{D}{d^2} = 2d \cdot \log_e \frac{D}{d} - d \quad . \quad (97)$$

For a maximum value we have

$$2d \cdot \log_e \frac{D}{d} = d \quad \text{or} \quad 2 \log_e \frac{D}{d} = 1$$

and therefore $$\log_e \frac{D}{d} = \tfrac{1}{2} = \log_e \sqrt{e} = \log_e e^{0.5} \quad . \quad . \quad . \quad (98)$$

i.e. the result is the logarithm to the base e of the square root of 2·71828.

Another example of the same type, but for a minimum value, is the following.

Problem 3.—What is the minimum value of

$$y = \frac{\sqrt{2A}}{d \log_e \frac{D}{d}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

This is solved by means of a graph in Fig. 4, if D=10 and A=½.

D is again a constant maximum and d a variable less than D. d is plotted along the horizontal line against y which is plotted vertically. It will be seen that the minimum value of y is obtained when $\log_e \frac{D}{d} = 1$. The number which has this logarithm is 2·71828, the base of the natural logarithms.

To solve by the calculus we have

$$\frac{dy}{dd} = \frac{-\sqrt{2A}\left(\log_e \frac{D}{d} - 1\right)}{\left(d \log_e \frac{D}{d}\right)^2} = \frac{\sqrt{2A}\left(1 - \log_e \frac{D}{d}\right)}{\left(d \log_e \frac{D}{d}\right)^2} \quad (100)$$

For a minimum value we have

$$\log_e \frac{D}{d} = 1, \text{ and therefore } \frac{D}{d} = 2{\cdot}71828 \quad . \quad . \quad (101)$$

General Remarks on Maxima and Minima Values.

In dealing with problems it is generally desirable to construct a curve such as Fig. 2, and not rely entirely on the calculus method. From the curve we are enabled to see not only the maximum or minimum point, but the relations between x and y (as in Fig. 2) for all the values in the graph. We sometimes learn, for example, that the values of x may vary very widely from the theoretical maximum or minimum value, whilst at the same time y changes very slowly, and this may have an important bearing on practical problems.

21. Simple Harmonic Motion and the Sine Function.—The simplest wave form of an alternating voltage or current is the sine wave, an example of which is shown in Fig. 5. In practice alternating voltage and current waves frequently differ considerably

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from the sine wave form, but by Fourier's theorem it may be shown that any wave of a given form other than a sine wave may be expressed as the sum of a series of sine waves. Sine functions

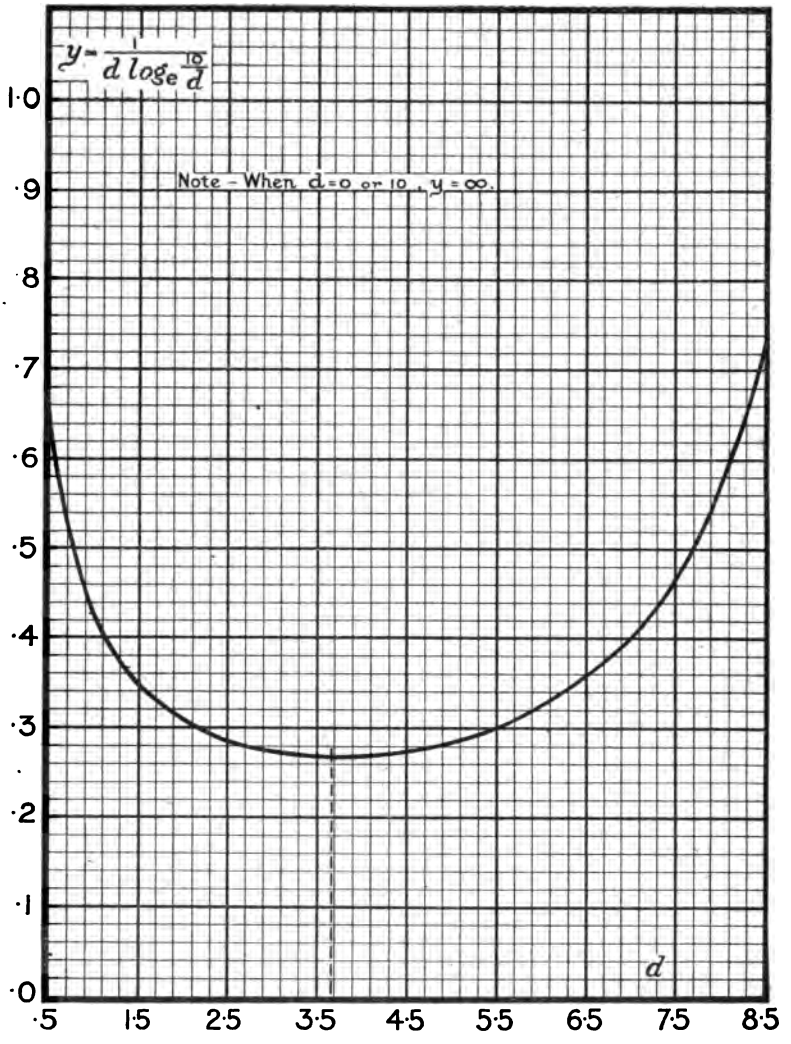

FIG. 4.

are extensively used in alternating current calculations, and are of great importance in the study of telephonic transmission. Some of the principal points in connection with them will, therefore, now be explained. If the radius OP in Fig. 5 rotates at a uniform

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speed in an anti-clockwise direction, commencing in the horizontal position OA, the vertical line ON, which is the projection of OP on the vertical diameter of the circle, will increase in length as OP rotates from OA until it arrives in the vertical position, when OP = ON. If OP rotates with a uniform velocity the increase of ON in a given small interval of time will differ considerably with its position, being much greater when OP is near OA than when it is near the vertical position. As OP moves through the second quadrant of the circle the apparent motion of N along the vertical diameter will be exactly reversed as compared with its motion in the first quadrant of the circle. As the radius rotates through the lower half of the circle the projection of OP on the lower half of the vertical diameter of the circle will be an exact repetition of the movement along ON in the upper part of the circle, but reversed

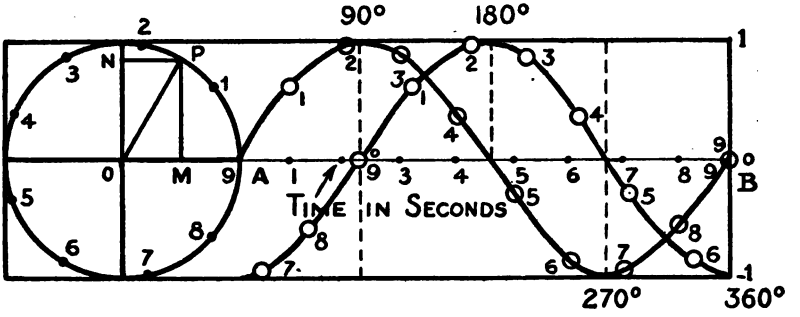

FIG. 5.

in direction, until finally OP completes its movement through the fourth quadrant and arrives at the starting point OA.

The apparent motion of N along the diameter with varying speed, due to the uniform motion of OP, is called Simple Harmonic Motion, and the length of ON at any instant is the sine of the angle ωt through which OP has revolved from OA, at the instant when ON is measured, if the length OP is 1. In any case $\sin \omega t = \frac{ON}{OP}$ whatever be the lengths of ON and OP. Let a line AB now be drawn at right angles to the vertical diameter of the circle, as a continuation of the horizontal diameter, and divided into equal parts such that the total parts represent the time taken by OP to make one revolution. If now a small dot or circle be plotted over each of the equal intervals of time shown on the horizontal line commencing with 1, the height of the dot or circle being equal to the length of ON, the projection of OP at that instant, *i.e.* to the sine of the angle swept through by OP and measured at the instant shown on the axis of time, then a curve drawn as shown through all the small plotted circles is a sine curve. If the intensity of an electric current varies at equal intervals of time in such a manner that the instantaneous values of the current when plotted give a curve such as that shown in Fig. 5, the current follows the sine law,

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and the same remark may apply to voltages impressed on an alternating current circuit.

22. Angular Velocity.—If the line OP in Fig. 5 makes one revolution per second it has an angular velocity of 2π radians, or 360 degrees per second, the radian is the usual angular unit, and the second the unit of time. If OP revolves f times per second, it has evidently an angular velocity of $2\pi f$ radians per second, and in t seconds it revolves through $2\pi ft$ radians. At the instant t the sine of the angle will be y, and

$$y = \sin wt \quad . \quad . \quad . \quad . \quad . \quad (102)$$

where $w = 2\pi$ times the frequency = the angular velocity in radians per second.

23. Amplitude.—The maximum value of ON, Fig. 5, is the maximum amplitude of the curve, and in (102) this is the maximum value of the sine which equals 1. The amplitude may, however, have any value; if this value be a (102) becomes

$$y = a \sin wt \quad . \quad . \quad . \quad . \quad . \quad (103)$$

The effect of this is that every ordinate in the sine curve in Fig. 5 is multiplied by a, seeing that the maximum ordinate was previously 1.

24. Phase, Lag and Lead.—Suppose that the sine curve, instead of starting from the end of the horizontal line, where it represents sin O at the commencement, begins at a point where it represents sin ϕ (see Fig. 6), where ϕ is an angle lagging behind O, then as OP (Fig. 5) revolves, the sine curve will have exactly the same form as before, since the circular motion, and therefore its projection ON, will still pass through the same cycle, the required value y of the sine of the angle at any instant t, however, is now

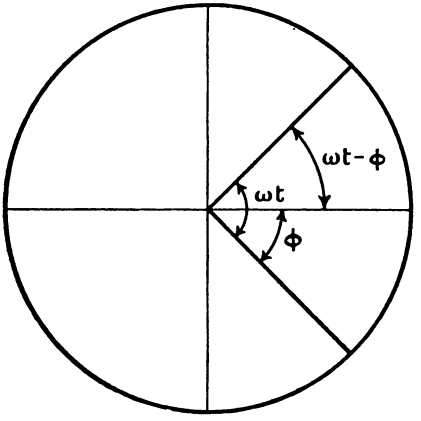

FIG. 6.

$$y = a \sin (wt - \phi) \quad . \quad (104)$$

i.e. the sine of an angle lagging behind wt by ϕ degrees. In Fig. 5 a sine curve is shown which lags by 90°.

The equation for the sine of an angle leading the angle wt by ϕ degrees is

$$y = a \sin (wt + \phi) \quad . \quad . \quad . \quad . \quad (105)$$

25. The Cosine Function.—A simple harmonic or periodic motion may also be represented by a cosine formula, for as OP revolves (Fig. 5) its projection OM on the horizontal diameter of the circle (where OP = 1) at any instant is the cosine of the angle through which OP has revolved after starting from the vertical position.

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26. The Equation of Simple Harmonic Motion.—If Equation (105) is differentiated with respect to the time, we obtain

$$\frac{dy}{dt} = wa \cos (wt + \phi) \quad . \quad . \quad . \quad . \quad (106)$$

differentiating a second time we have

$$\frac{d^2y}{dt^2} = -w^2a \sin (wt + \phi) \quad . \quad . \quad . \quad (107)$$

Substitute y for $a \sin (wt + \phi)$ from (105) and we obtain

$$\frac{d^2y}{dt^2} = -w^2y \quad . \quad . \quad . \quad . \quad . \quad (108)$$

If any equation of this type is met with it will be recognized as an equation of the harmonic type. It should be noted that w is a constant and y a variable.

27. Length and Time in Sine Functions.—In Fig. 5 the time t is represented by equal distances along the axis, and these equal distances correspond to equal angles measured round the circle when OP revolves at a uniform speed; the sine of the angle is in fact a function of the time and of the distance along AB.

If y represents an alternating current of sine form which is propagated along a transmission line as shown in Fig. 7, the relation between the time and the distance may be seen for the particular case which has been calculated. If in such an instance the angle revolved through by the current per unit length is α, then at a distance of l miles from the sending end of the circuit the angle is αl, and the sine of the angle represented by the large curve is

$$y = \sin \alpha l = \sin wt \quad . \quad . \quad . \quad . \quad (109)$$

In Equation (105) the angle αl may be substituted for wt. In the case of (109) the angle ϕ has been neglected for simplicity of illustration.

28. Wave Length and Wave Length Constant.—It will be seen from Fig. 7 that a complete period is shown between the points 0 and 4 on the time axis, and that the point 4 is at a distance of 236 miles from the transmitter. A complete period is equal to 2π radians and the complete cycle has been revolved through in 236 miles; the change per mile is therefore $\frac{2\pi}{236} = 0{\cdot}0266$ radian. The amount by which the angle changes per mile is called the wave length constant. If this be called α, then the connection between the wave length l the angle α and 2π is

$$\frac{2\pi}{\alpha} = l \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Referring again to Fig. 7. Since $\frac{4}{3000} = \frac{1}{750}$ second is the time of one period we have evidently 750 cycles per second, and at any instant t we have

$$y = \sin 2\pi \times 750t = \sin \alpha l \quad . \quad . \quad . \quad (111)$$

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As an illustration of the connection between time and length, take the value of y at the point 16 on the axis of time, 940 miles from the sending end. We have

$$(a)\quad y = \sin 2\pi \times 750 \times \tfrac{16}{3000} = \sin 2\pi \times 4 = \sin 25{\cdot}1$$

i.e. the wave has revolved through four complete cycles, and as the same values of the sine recur in each cycle, this is the same as 2π

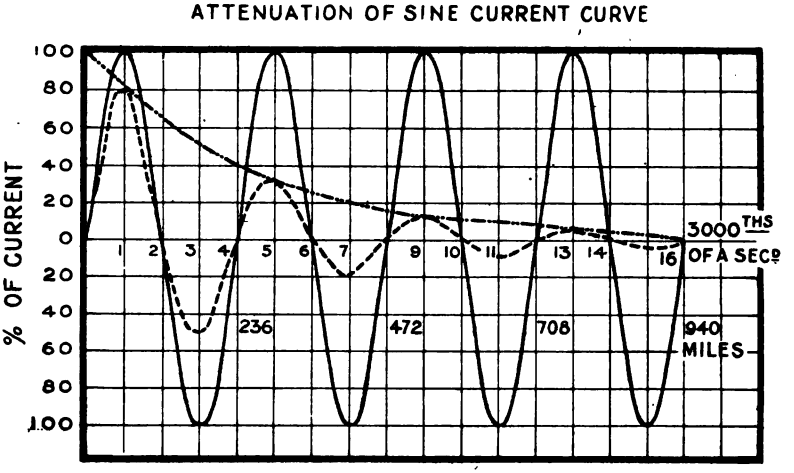

FIG. 7.

or 6·28 radians, and is further the same as sin 0, therefore y in this case $= 0$.

$$(b)\quad y = \sin al = \sin 0{\cdot}0266 \times 940 = \sin 25 \quad . \quad (112)$$

which is practically the same value as given by case (a), but expressed in terms of the angle a and the length l instead of in terms of frequency and time.

29. Attenuation of the Sine Wave.—Fig. 7 represents a case in which the sine wave is subject to a continual attenuation according to the exponential law, and the attenuated sine wave is shown in dotted lines. The curve connecting the maxima values of the sine wave shows the attenuation of the current at any length on the line; this value may be calculated by the equation

$$y = 100e^{-\beta l} \quad . \quad . \quad . \quad . \quad . \quad (113)$$

If the maximum amplitude of the current at the sending end be represented by 100, and if the attenuation constant $\beta = 0{\cdot}00395$, then y gives the maximum amplitude of the wave at any point l. If phase be included in the calculation, we have

$$y = 100e^{-\cdot00395l} \sin 0{\cdot}0266l$$

The law governing such cases may be expressed in general terms thus

$$y = Ae^{-\beta l} \sin (al + \phi) \quad . \quad . \quad . \quad . \quad (114)$$

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This may be written in the form

$$y = e^{-\beta l}(b \cos \alpha l + a \sin \alpha l) \quad . \quad . \quad . \quad (115)$$

as will now be proved.

Referring to Fig. 1, where $A = \sqrt{a^2 + b^2}$, we have $a = A \cos \phi$, and $b = A \sin \phi$. Substitute these values in (115) and we obtain

$$\begin{aligned} y &= e^{-\beta l}(A \sin \phi \cos \alpha l + A \cos \phi \sin \alpha l) \\ &= Ae^{-\beta l} \sin (\alpha l + \phi) \end{aligned}$$

which is the same as (114).

From this it is clear that if the values of a and b are suitably chosen (114) = (115). In this case $A = \sqrt{a^2 + b^2}$, and $\tan \phi = \frac{b}{a}$

If the angle lags we have

$$y = Ae^{-\beta l} \sin (\alpha l - \phi) \quad . \quad . \quad . \quad . \quad (116)$$

If $a = A \cos \phi$ and $b = -A \sin \phi$, this becomes

$$y = e^{-\beta l}(A \sin \alpha l \cos \phi - A \sin \phi \cos \alpha l)$$

or

$$y = Ae^{-\beta l} \sin (\alpha l - \phi)$$

So that both (114) and (116) may be expressed in the general form of equation (115).

Any equation of the form

$$y = Ae^{j\theta} + Be^{-j\theta} \quad . \quad . \quad . \quad . \quad . \quad (117)$$

may also be expressed in the form of Equation (115), as will now be shown.

Equation (117) is the sum of Equations (7) and (8) with the coefficients changed. In order to make (117) as general as possible, let A and B be vectors such that $A = A_1 + jA_2$ and $B = B + jB_2$, then from (7) and (8) we may write

$$y = a + jb = (A_1 + jA_2)(\cos \theta + j \sin \theta) + (B_1 + jB_2)(\cos \theta - j \sin \theta) \quad (118)$$

Equating the real and imaginary parts of (118), we have

$$a = (A_1 + B_1) \cos \theta - (A_2 - B_2) \sin \theta \quad . \quad . \quad (119)$$

$$b = (A_2 + B_2) \cos \theta + (A_1 - B_1) \sin \theta \quad . \quad . \quad (120)$$

As the coefficients in both these equations are not restricted as regards value, each of them may be written in the form of Equation (115).

30. Distinction between Transient Voltages and Currents, and those which have attained the Steady State.—Equations (117), (118), etc., relate to harmonic voltages and currents, as already stipulated. When, however, an harmonic alternating current starts in a circuit it is usually accompanied at the commencement by a transient wave which dies away exponentially and quicker than the harmonic current. It is generally assumed that this transient stage in telephony is so short as to be negligible (see Chap. V.), and that the speech currents attain the steady harmonic state. The transient

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stage is therefore not taken into account in the foregoing investigation. It is dealt with in Prof. Berg's "Electrical Engineering," chap. ix. (McGraw Hill Book Co., 1916). See also Malcolm's "Theory of the Submarine Telegraph and Telephone Cable," chap. viii., The Electrician Co., 1917, etc.

31. Titles of Tables of Hyperbolic Functions.—Transmission calculations are very much facilitated by the use of tables of hyperbolic functions, and the following will be found extensive and reliable:—

(*a*) "Smithsonian Mathematical Tables," "Hyperbolic Functions," prepared by George F. Becker and C. E. Van Orstrand. These tables are published by the Smithsonian Institution, City of Washington.

(*b*) "Tables of Complex Hyperbolic and Circular Functions," together with the accompanying "Chart Atlas of Complex Hyperbolic and Circular Functions," by A. E. Kennelly, Sc.D., A.M., Harvard University Press.

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CHAPTER II

THE INFINITE LINE—DIRECT CURRENT CASE

32. General Remarks.—The study of Telephonic Transmission is the study of the phenomena associated with the telephone speaking circuit, and more particularly of that part of it which deals with the propagation of the electric currents generated by the human voice. The electric currents generated by speech are alternating currents; the laws governing their propagation are, however, exactly the same as those for direct currents, and on account of the relatively simple character of the latter, there is a great advantage in first applying the laws of transmission to the direct current case. Such a study has, moreover, a direct and important application. All the formulæ so developed are suitable for application to overhead telegraph lines, and, indeed, to any lines having uniformly distributed resistance and leakance, in cases where the currents have attained the steady state. This applies, for example, to such commonplace operations as the measurement of the conductor and insulation resistance of long lines having uniformly distributed leakance and resistance. At present approximate formulæ are often applied to telegraph lines in such cases, and in certain instances these formulæ are far from having the accuracy of those here developed. The chapters relating to direct current transmission have, therefore, a special value for telegraph engineers, in addition to the fact that the methods here developed provide the simplest possible introduction to the subject of Telephonic Transmission.

The formulæ for the infinite line are, generally speaking, less complicated and more easily understood than those for finite lines, and as may be expected, all formulæ for finite lines reduce to those for infinite ones if the circuit is of sufficient electrical length. From this point of view the infinite line case is only a limiting one for the more general case coming under the heading of finite lines. In practical telephonic transmission, however, the conception of the infinite line is much used and is of great importance. This is not merely because many chains of telephonic circuits are sufficiently long to approximate to infinite line conditions, but also because in the interpretation of the effects of reflection considerable use is made of the conception of the infinite line, as will be seen later (see Chap. VII.).

33. Formulæ for Current and Volts in the Infinite Line.—The general mathematical law relating to such circuits is proved in Appendix I., and in a more elementary manner in Appendix II., and the subject is here further developed. Generally speaking, very long mathematical proofs in this book are given in appendixes,

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and it is assumed that the relative appendix has been read. It follows from the appendixes named that the infinite line is a circuit in which the volts transmitted over the line continually meet an unvarying resistance (R_0) called the characteristic resistance, and at the same time the current in the circuit is subject to a continuous and uniform leakance. The result of these combined actions is that both the volts and the resulting current are continuously attenuated as they progress along the line, and their amplitude is reduced in each case in accordance with the compound interest law (see Fig. 7, Chap. I.). When this law is fulfilled the volts E_2 and the current I_2 at any point l on the line, are expressed by the following equations (see App. I. and II. for proofs):—

$$E_2 = E_1 e^{-\gamma l} = E_1 e^{-\theta} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$I_2 = \frac{E_1}{R_0} e^{-\gamma l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence

$$\frac{E_2}{E_1} = e^{-\gamma l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\frac{I_2}{I_1} = e^{-\gamma l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

E_1 is the impressed voltage and I_1 the current at the sending end of the line, also $\gamma l = \theta$.

34. The Attenuation Constant and Characteristic Resistance.—In the continuous current case there is no difference between the propagation constant γ and the attenuation constant β, and the symbol γ is retained for the attenuation constant. We have by Appendix I.—

$$\gamma = \sqrt{RG} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This formula gives the attenuation constant of a circuit having uniformly distributed resistance and leakance.

The characteristic resistance in the direct current case is, by Appendix I.—

$$R_0 = \sqrt{\frac{R}{G}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

35. Relation between γ, R_0, R and G.—When the attenuation constant and characteristic impedance are known, R and G may readily be deduced from them as follows:—

$$\gamma \times R_0 = \sqrt{RG} \times \sqrt{\frac{R}{G}} = \sqrt{R^2} = R \quad . \quad . \quad (7)$$

$$\frac{\gamma}{R_0} = \frac{\sqrt{RG}}{\sqrt{\frac{R}{G}}} = \sqrt{G^2} = G \quad . \quad . \quad . \quad . \quad (8)$$

The law which finds its mathematical expression in (1) and (2) is of fundamental importance, as by its aid a large number of other useful transmission formulæ may be derived, and its proper interpretation affords an explanation of some of the most important

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phenomena of transmission. γ and R_0 are constants which enter largely into transmission formulæ, and they are sometimes called the characteristic constants of the line, R and G being the secondary constants.

36. Approximations to the Infinite Line Condition.—As explained in Appendix II., however, an infinite line taken in the literal sense does not exist; but a line with uniformly distributed constants, and of any length, fulfils the same law, if the receiving end of the line is closed through a resistance R_0 the same as that of the characteristic resistance of the line.

Any electrically long circuit, however, fulfils approximately the condition $2\frac{E_1 e^{-\gamma l}}{R_0} = I_2$ (see Chap. III., Fig. 15, and remarks on that figure) when short-circuited at the receiving end of the line. If any two circuits are sufficiently long, therefore, their comparative value is approximately as follows:—

$$\left.\begin{aligned} 2\frac{E_1}{R_0} e^{-\gamma l} &= I_2 \\ 2\frac{E_1}{R'_0} e^{-\gamma_1 l_2} &= I'_2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It will be convenient to call such circuits "semi-infinite circuits."

In these equations let γ and l be the attenuation constant and length of one circuit and γ_1 and l_2 the attenuation constant and length of the other. R_0 and R'_0 are the corresponding characteristic resistances. If the circuit lengths are such that the received currents are equal, we have approximately

$$2\frac{E_1}{R_0} e^{-\gamma l} = 2\frac{E_1}{R'_0} e^{-\gamma_1 l_2}$$

whence

$$\frac{e^{-\gamma l}}{R_0} = \frac{e^{-\gamma_1 l_2}}{R'_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now if the two lines be extended to an infinite length their comparative value would be the same as that given by this equation, as will be seen by comparison with (2). It therefore follows that the comparison of two sufficiently long circuits leads to the same result whether the lines be infinite lines or not. Considerable use is made of this fact in standard cable measurements, as will be seen in Chap. XI. For such measurements circuits having a βl less than 1 are often sufficiently long (see Chap. VII. under "Long and Short Transmission Lines").

For purposes of machine measurements a line may be said to be electrically long if it has a value of $\gamma l = 2$. If the line is increased in length beyond this the approximation to an infinite line continually becomes closer; but unless the circuit is closed through a resistance R_0 or is indefinitely long, the received current *never* exactly agrees with that at the same point in an infinite line. By making the line sufficiently long, however, the difference in the two cases made be made as small as we please (see Fig. 15, Chap. III.).

37. Relative Effect of γ and R_0 on the Received Current.—If

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apparatus is added at the sending end of the line the voltage impressed on the infinite line will be reduced, and the equation connecting volts and received current may be written

$$\frac{E_1}{R_a + R_0} e^{-\gamma l} = I_2 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It is sometimes convenient to express R_0 and other impedances or resistances in the exponential form. In such a case let $R_0 = e^q$, then (2) may be written

$$\frac{E_1}{e^q} e^{-\gamma l} = E_1 e^{-q} \times e^{-\gamma l} = E_1 e^{-(\gamma l + q)} = I_2 \quad . \quad (12)$$

From this it may be seen that if q is relatively large in comparison with γl, then the magnitude of q will largely affect the magnitude of the received current I_2; on the other hand, if q is very small in comparison with γl, the effect of q on the received current will be small; this is more likely to be the case in an electrically long line, since the longer the line the greater will be the value of γl.

If the leakance G is large and the resistance R small the effect of R_0 will be relatively small and vice versa.

88. Comparison of the Transmission Efficiency of two Infinite Circuits.—If two circuits have the same characteristic resistance R_0, but have propagation constants of different value γ and γ_1, and if the lengths along the infinite line are l and l_1 respectively (or if the circuits have the total lengths l and l_1 and are each closed by a resistance R_0), the lengths being such that equal received currents I_2 are observed, then the relations between γ, γ_1 and l, l_1 are

$$\frac{l_1}{l} = \frac{\gamma}{\gamma_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This may be proved as follows for equal received currents in the two cases stipulated.

From (2) we have

case (*a*) $$\frac{E_1}{R_0} e^{-\gamma l} = I_2 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

case (*b*) $$\frac{E_1}{R_0} e^{-\gamma_1 l_1} = I_2 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

whence $$e^{-\gamma l} = e^{-\gamma_1 l_1} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and $$\gamma l = \gamma_1 l_1 \quad \text{that is} \quad \frac{l_1}{l} = \frac{\gamma}{\gamma_1} \quad . \quad (17)$$

which is the same as (13).

This result may be defined thus—

The relative transmission efficiency of any two infinite circuits having the same characteristic impedance, is measured by the relative lengths of the two circuits through which equal currents are received with equal values of impressed voltage. In such a case the relative transmission efficiency of the two circuits varies inversely as their attenuation constants.

If the values of R_0 are not the same in the two circuits to be

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compared, we have, for equal received currents, if R_0, γ, l and γ_1 remain unaltered, and l_2 is varied to meet the altered value R'_0,—

$$\frac{E_1}{R_0} e^{-\gamma l} = I_2$$

$$\frac{E_1}{R'_0} e^{-\gamma_1 l_2} = I_2$$

whence

$$\frac{e^{-\gamma l}}{R_0} = \frac{e^{-\gamma_1 l_2}}{R'_0} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If $R_0 = e^q$ and $R'_0 = e^{q_1}$, we get

$$e^{-(\gamma l + q)} = e^{-(\gamma_1 l_2 + q_1)} \quad \text{whence} \quad \gamma l + q = \gamma_1 l_2 + q_1 \quad (19)$$

The length of circuit required to give equal transmission is now l_2 and is evidently not the same as in (13). (13) and (18) have an important application in standard cable measurements.

It is evident from the foregoing considerations that the characteristic impedance must be carefully taken into account in dealing with the relative and absolute transmission efficiency of circuits.

39. Power in the Infinite Line.—An inspection of (1) and (2) shows that both volts and current are attenuated in accordance with the same law. To find the power P at any point on the line, we have

$$P = E_1 e^{-\gamma l} \times \frac{E_1}{R_0} e^{-\gamma l}$$

that is

$$P = \frac{E_1^2}{R_0} e^{-2\gamma l} = \frac{E_1^2}{R_0} e^{-2\theta} \quad . \quad . \quad . \quad . \quad (20)$$

where $\gamma l = \theta$.

The power at the sending end is therefore $\frac{E_1^2}{R_0}$, and it is attenuated as it progresses along the infinite line directly as the square of the attenuation factor $e^{-\theta}$. The amount of the attenuation for a given length l depends on the value of $\gamma = \sqrt{RG}$.

The losses due to both R and G are energy losses, and they can only be reduced by decreasing the values of R or G, or both. It is evident that energy is lost by the fact that a portion of it is spent in heating the conductor, and this part varies as the square of the current. In the case of G, this symbol gives the magnitude of a direct loss of current to earth, which involves a loss of energy. An analysis of the formulæ will show that a reduction in R decreases both the attenuation constant and the characteristic impedance, whereas a reduction of G decreases γ and increases R_0.

40. Comparison of the Relative Power in Two Infinite Circuits.—If two circuits have the same voltage E_1 impressed upon them, and have the same characteristic impedance R_0, but have attenuation constants of different values γ and γ_1, and further, if the lengths of the two circuits l and l_1 are such as to give equal attenuation lengths, as in (13), then

$$P = P_1$$

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Proof.—From (1) and (2) we have the comparative power P and P_1; we then obtain

$$\frac{P}{P_1}=\frac{E_1e^{-\gamma l}\times\frac{E_1}{R_0}e^{-\gamma l}}{E_1e^{-\gamma_1 l_1}\times\frac{E_1}{R_0}e^{-\gamma_1 l_1}}=\frac{\frac{E_1^2}{R_0}e^{-2\gamma l}}{\frac{E_1^2}{R_0}e^{-2\gamma_1 l_1}}$$

or

$$\frac{P}{P_1}=\frac{e^{-2\gamma l}}{e^{-2\gamma_1 l_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

But for equal attenuation lengths (attenuation constant × length) we have

$$e^{-\gamma l}=e^{-\gamma_1 l_1}$$

and therefore

$$e^{-2\gamma_1 l_1}=e^{-2\gamma l}$$

and therefore in (21)

$$P=P_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

It follows that if $\frac{l_1}{l}=\frac{\gamma}{\gamma_1}$, which is the result given in (13), then the comparative transmission efficiency is the same for volts, current, or power in this case.

If the characteristic resistances are not equal, then if the symbols γl, R_0, R'_0 and γ_1 are the same as before and l_3 be varied to obtain equality of energy, we have

$$1=\frac{P}{P}=\frac{E_1e^{-\gamma l}\times\frac{E_1}{R_0}e^{-\gamma l}}{E_1e^{-\gamma_1 l_3}\times\frac{E_1}{R'_0}e^{-\gamma_1 l_3}}=\frac{\frac{E_1^2}{R_0}e^{-2\gamma l}}{\frac{E_1^2}{R'_0}e^{-2\gamma_1 l_3}}$$

$$=\frac{R'_0}{R_0}\frac{e^{-2\gamma l}}{e^{-2\gamma_1 l_3}} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If $R_0=e^q$ and $R'_0=e^{q_1}$ (23) becomes

$$1=\frac{P}{P}=\frac{e^{-(2\gamma l+q)}}{e^{-(2\gamma_1 l_3+q_1)}} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Whence $2\gamma l+q=2\gamma_1 l_3+q_1$, since P is the same in each case.

A comparison with (10) will show that in this case the comparative power at the receiving end of a semi-infinite line is not the same as the comparative attenuation length of the volts and of the current taken separately.

If apparatus is added to the circuit all the results given are modified: this branch of the subject is, however, more conveniently dealt with at a later stage (Chaps. VI. and VII.).

The following examples have been worked out to illustrate the formulæ already developed.

Example 1.—A circuit of infinite length has a resistance of 25·14 ohms per mile loop and a leakance of 10^{-6} mhos wire to wire per mile loop. If 10 volts is applied to the sending end of the circuit, find

I. The propagation constant γ.
II. The characteristic resistance R_0.
III. The length of circuit l and the received current I_2 when (*a*) $\gamma l=1$, and (*b*) $\gamma l=2$.

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I. From (5) $\gamma = \sqrt{RG}$
$= \sqrt{25{\cdot}14 \times 10^{-6}}$
$= 5{\cdot}014 \times 10^{-3}$ hyps. per mile loop

II. From (6) $R_0 = \sqrt{\frac{R}{G}}$
$= \sqrt{25{\cdot}14 \times 10^{6}} = 5014$ ohms

III. (*a*) If $\gamma l = 1$, then from Ex. 1, $l = \frac{10^3}{5{\cdot}014} = 199{\cdot}44$ miles.

And by substitution in (2)

$$I_2 = \frac{10e^{-1}}{5014} = 0{\cdot}7337 \times 10^{-3} \text{ amps.}$$

(*b*) If $\gamma l = 2$, then $l = \frac{2 \times 10^3}{5{\cdot}014} = 398{\cdot}88$ miles;

and $$I_2 = \frac{10e^{-2}}{5014} = 0{\cdot}2699 \times 10^{-3} \text{ amps.}$$

Example 2.—In order to show the effect of the characteristic resistance in infinite lines, tabulate similar details for a conductor having a loop resistance of 44 ohms per mile, and for one having a loop resistance of 11·73 ohms per mile, the leakance in each case being 10^{-6} mhos per mile.

The results are given in Table I.

TABLE I.

Conductor constants per mile loop.		γ in hyps. per mile loop.	R_0 in ohms.	Length of circuit l miles when		Received current I_2 milli-amps. when	
R ohms.	G mhos.			$\gamma l = 1$	$\gamma l = 2$	$\gamma l = 1$	$\gamma l = 2$
44·00	10^{-6}	·006633	6633	150·76	301·51	0·5546	·2040
25·14	10^{-6}	·005014	5014	199·44	398·88	0·7337	·2699
11·73	10^{-6}	·003425	3425	291·98	583·96	1·0741	·3951

An inspection of the table will show that equal currents are not received when $\gamma l = \gamma_1 l_1$, as would be the case if the corresponding characteristic resistances were equal: see (17). In such a case all the received currents under the heading $\gamma l = 1$ would be equal, and this remark would also apply to the currents under the heading $\gamma l = 2$. Such is, however, far from being the case, and the difference is entirely due to the differences in the corresponding characteristic resistances.

It should be noted that if any two infinite circuits, which are to be compared, have the same leakance G, then

$$\frac{R_0}{R'_0} = \frac{\gamma}{\gamma_1}$$

because $$\frac{\gamma}{\gamma_1} = \sqrt{\frac{RG}{R_1 G}} = \sqrt{\frac{R}{G}} \times \sqrt{\frac{G}{R_1}} = \frac{R_0}{R'_0}$$

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CHAPTER III

THE EQUIVALENT CIRCUIT, DIRECT CURRENT CASE

41. Introductory Remarks.—The term "Equivalent Circuit" is here used to describe an artificial circuit which is equivalent for all electrical operations at one frequency, conducted from its two ends, to a real circuit having uniformly distributed electrical constants. In the alternating current case it is necessary in most instances to use resistance, capacity, and inductance to correctly make up an equivalent circuit, which at a given definite frequency is equivalent to a uniform transmission line; but in the direct current case the equivalent circuit only requires a combination of three resistances to correctly represent a uniform line, having distributed resistance and leakance. This case will now be dealt with. Artificial circuits having the properties here outlined are dealt with by Dr. Breisig ("Theoretische Telegraphie," p. 317), and very extensively by Dr. Kennelly in "The Application of Hyperbolic Functions to Electrical Engineering Problems," and by others. Generally speaking it will be convenient to represent such circuits as single wire circuits, seeing that their study leads to the same results and formulæ as double wire circuits. This is a well recognized method, and one which gives correct results, and is, moreover, simpler for purposes of study than the double wire circuit. Equivalent circuits made up of three resistances are of two kinds, viz. (*a*) the T or star circuit as shown in Fig. 9, and (*b*) the Π (Pi) or delta circuit as shown in Fig. 20. A double wire circuit is represented in Fig. 11. This is sometimes known as an H circuit, and sometimes as an I circuit. The uniform circuit corresponding to the equivalent circuits is shown in Fig. 8. This real circuit has a propagation constant $\gamma=\sqrt{RG}$, and a characteristic resistance $Z_0=\sqrt{\frac{R}{G}}$. (The symbol Z_0 will be used for the characteristic resistance henceforth for convenience and in anticipation of the alternating current case.) We also have $\gamma l=\theta$ the hyperbolic angle of the line. It is proved in an elementary manner in Appendixes III. and IV. that the equivalent circuit described may consist either of a T or a Π circuit, and some simple relations between these circuits are given at the end of this chapter. It will be convenient to use the T circuit for our immediate purpose—

For comparison with the equivalent circuit, the nominal circuit is shown in Fig. 10. This circuit is obtained by taking the total resistance Rl of the uniform circuit, as shown, and placing a leak containing the total circuit leakance Gl at its centre. By comparison with Fig. 9 the error of lumping as in Fig. 10 in any given case may be found.

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The equivalent circuit, as is well known, offers an excellent means of explaining the action of lumped loading and artificial circuits, and for that reason alone its inclusion in a work of this kind would be desirable. It will, however, now be demonstrated that the equivalent T circuit offers a relatively simple means of developing a number of important transmission formulæ relating to uniform circuits.

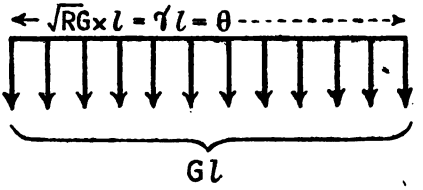

FIG. 8.—Uniform line.

The method of developing the required formulæ is as follows. Since $R_2 = Z_0 \tanh \frac{\theta}{2}$ and $R_1 = \frac{Z_0}{\sinh \theta}$, R_1 and R_2 being independent components of the T circuit (Appendix III.), any combination whatever of R_1 and R_2 (such as the sum, the product, or the square of these numbers, for example) will be equal to some corresponding combination of $Z_0 \tanh \frac{\theta}{2}$ and $\frac{Z_0}{\sinh \theta}$. It is proved in Appendix III. that although R_1 and R_2 are components of an artificial line, the hyperbolic functions here given are the corresponding values applying to a real uniform transmission line.

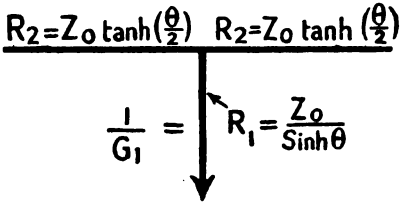

FIG. 9.—The equivalent T circuit.

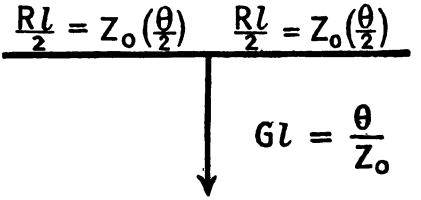

FIG. 10.—The nominal T circuit.

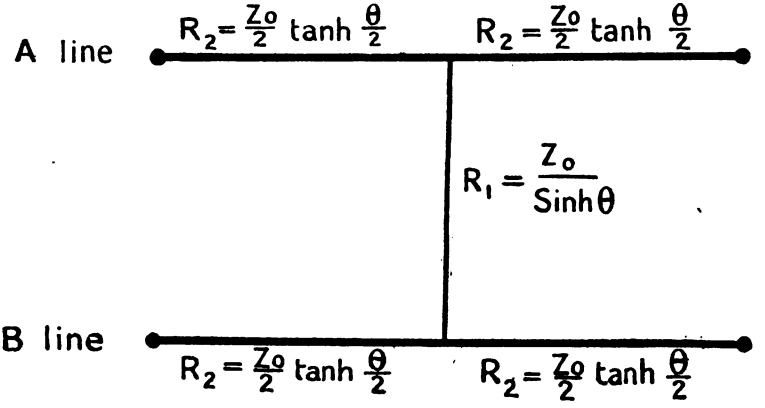
A line
$R_2 = \frac{Z_0}{2} \tanh \frac{\theta}{2}$
$R_2 = \frac{Z_0}{2} \tanh \frac{\theta}{2}$
$R_1 = \frac{Z_0}{\sinh \theta}$
B line
$R_2 = \frac{Z_0}{2} \tanh \frac{\theta}{2}$
$R_2 = \frac{Z_0}{2} \tanh \frac{\theta}{2}$

FIG. 11.—The equivalent H circuit.

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It follows, therefore, that a number of equations may be written in terms of R_1 and R_2 (by the aid of the equivalent T diagram), and it can be shown that the equivalent form of the equations in hyperbolic functions, which holds good for the real line can be arrived at by means of these equations without difficulty, and that in all cases the formulæ so obtained are the same as those developed by the usual method in Appendix I. for the uniform circuit. The fact that a given formula is expressed as a simple expression according to Ohm's law on the one hand, and in hyperbolic functions on the other hand, affords a concrete example of the meaning of the hyperbolic function, and the examples at the same time demonstrate the utility and advantage of the latter. In the process of deriving these formulæ it will be advantageous to refer to the following list.

Equivalent Formulæ for Reference.—

$$R_2 = Z_0 \frac{(e^\theta - 1)}{(e^\theta + 1)} = Z_0 \tanh \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad (1)$$

$$R_1 = \frac{2Z_0 e^\theta}{(e^\theta + 1)(e^\theta - 1)} = \frac{Z_0}{\sinh \theta} \quad . \quad . \quad . \quad (2)$$

$$R_2^2 + 2R_1R_2 = Z_0^2 \quad . \quad . \quad . \quad . \quad (3)$$

$$R_1 + R_2 = Z_0 \coth \theta \quad . \quad . \quad . \quad (4)$$

The first three equations have already been proved in Appendix III., the proof of the fourth is as follows :—

$$R_1 + R_2 = Z_0 \frac{(e^\theta - 1)}{(e^\theta + 1)} + Z_0 \frac{2e^\theta}{(e^\theta + 1)(e^\theta - 1)} \text{ by (1) and (2)}$$

If we write a for e^θ we have

$$R_1 + R_2 = Z_0 \frac{(a - 1)}{(a + 1)} + Z_0 \frac{2a}{(a + 1)(a - 1)}$$

that is

$$Z_0 \left\{ \frac{(a-1)^2 + 2a}{(a^2 - 1)} \right\}$$

$$= Z_0 \left\{ \frac{a^2 + 1}{a^2 - 1} \right\}$$

$$= Z_0 \left\{ \frac{a + \frac{1}{a}}{a - \frac{1}{a}} \right\}$$

$$= Z_0 \left\{ \frac{(e^\theta + e^{-\theta})}{(e^\theta - e^{-\theta})} \right\}$$

$$= Z_0 \coth \theta$$

In order to show how formulæ (1) to (4) may be utilized in the development of transmission formulæ we will now apply them to the formation of some well-known mathematical expressions in connection with finite transmission lines, thus—

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42. To find the Sending-End Resistance of a Transmission Line of any Length Open at the Receiving End.—The diagram of the equivalent T circuit is shown in Fig. 12. By inspection of the figure the sending-end resistance of the equivalent circuit is $R_1 + R_2$, since R_1 is the only leak, and therefore by (4) the equivalent value in hyperbolic functions is immediately obtained, for

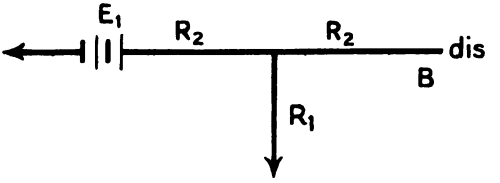

FIG. 12.

$$Z = R_1 + R_2 = Z_0 \coth \theta \quad . \quad . \quad . \quad . \quad (5)$$

The relations between $R_1 + R_2$ and $Z_0 \coth \theta$ were investigated in para. 41. Equation (4) is, however, placed in the general list given in para. 41 because it is frequently applied later.

43. Sending-End Resistance of a Transmission Line of any Length closed at the Receiving End.—Fig. 13 shows the equivalent T circuit

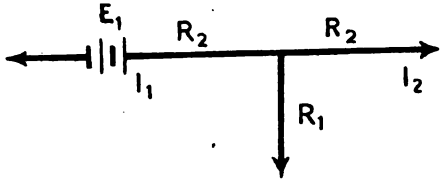

FIG. 13.

The total resistance of this circuit from the sending end E_1 is by inspection

$$Z_1 = R_2 + \frac{R_1 R_2}{R_1 + R_2}$$

After clearing of fractions the rearranged expression may be equated to an equivalent expression in hyperbolic functions by the aid of (3) and (4), thus—

$$Z_1 = \frac{2R_1R_2 + R_2^2}{R_1 + R_2} = \frac{Z_0^2}{Z_0 \coth \theta} = Z_0 \tanh \theta \quad . \quad (6)$$

By the aid of (5) and (6) an important proposition may be proved as follows:—

44. To prove that $\sqrt{Z \times Z_1} = Z_0$.—Multiplying the left-hand side of (5) by the left-hand side of (6), we have—

$$Z \times Z_1 = (R_1 + R_2)\frac{(2R_1R_2 + R_2^2)}{R_1 + R_2} = (2R_1R_2 + R_2^2) = Z_0^2 \text{ (by 3)}$$

Whence

$$\sqrt{ZZ_1} = Z_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Z_0 is the resistance of an infinite line.

The hyperbolic functions might have been applied directly to prove the formula in this case, but the resistances are utilized to illustrate the method.

Equations (5) to (7) give expressions for the resistance at the sending end of a transmission line without apparatus. In order to obtain the current at the sending end in such a case it is only necessary to remember that $\frac{E}{Z} = I$ and proceed as follows:—

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45. Current at the Sending End in a Transmission Line of any Length open at the Receiving End.—We have by (5), which gives the value of Z,

$$I = \frac{E_1}{Z} = \frac{E_1}{Z_0 \coth \theta} = \frac{E_1}{Z_0} \tanh \theta \quad . \quad . \quad . \quad (8)$$

46. Current at the Sending End in a Transmission Line of any Length closed at the Receiving End.—We have $I_1 = \frac{E_1}{Z_1}$, whence from (6)

$$I_1 = \frac{E_1}{Z_0 \tanh \theta} = \frac{E_1}{Z_0} \coth \theta \quad . \quad . \quad . \quad . \quad (9)$$

The current at the sending end of an infinite line whose resistance is Z_0, is $I_1 = \frac{E_1}{Z_0}$.

47. Current at the Receiving End in a Transmission Line of any Length closed at that End.—Fig. 13 applies in this case, which is more complicated than the preceding ones, owing to the fact that, as shown by (6), the sending-end resistance of the line changes as θ increases, and this has to be taken into account. Referring to Fig. 13, the current I_1 divides at the junction of R_1 and R_2, and the current I_2 flowing through R_2 to earth is $I_1 \times \frac{R_1}{R_1 + R_2}$. Substitute the value of I_1 from (9), and we get

$$I_2 = \frac{E_1}{Z_0} \coth \theta \times \frac{R_1}{R_1 + R_2}$$

Substitute the value of R_1 and $R_1 + R_2$ from (2) and (4) in the equation, and we obtain

$$I_2 = \frac{E_1 \coth \theta}{Z_0} \frac{Z_0 \sinh \theta}{Z_0 \coth \theta} = \frac{E_1}{Z_0} \coth \theta \times \frac{Z_0}{\sinh \theta} \times \frac{1}{Z_0 \coth \theta}$$

that is

$$I_2 = \frac{E_1}{Z_0 \sinh \theta} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the transmission line in question had no leakance the current I_2 at the receiving end would evidently be $I_2 = \frac{E_1}{R}$, where R is the ohmic resistance of the circuit. In (10) R is replaced by $Z_0 \sinh \theta$, and this is true for any length of line, so that $Z_0 \sinh \theta$ in such a case may be regarded as the receiving-end resistance of a transmission line of any length. A conventional proof of this is as follows:—

48. Resistance at the Receiving End of a Finite Transmission Line closed at the Receiving End.—The ratio of the impressed volts at the sending end to the received current is called the "Receiving-end Resistance," hence we have $\frac{E_1}{I_2} = Z_2$.

Whence by (10) $Z_2 = \frac{E_1}{\frac{E_1}{Z_0 \sinh \theta}} = Z_0 \sinh \theta \quad . \quad . \quad . \quad (11)$

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49. To find the Ratio $\frac{I_2}{I_1}$ in a Transmission Line of any Length.—This ratio is often very useful, and it is obtained as follows. By inspection of Fig. 13 the current at I_2 is

$$I_2 = I_1 \frac{R_1}{R_1 + R_2} \quad \therefore \frac{I_2}{I_1} = \frac{R_1}{R_1 + R_2}$$

By substitution from (2) and (4) this becomes

$$\frac{I_2}{I_1} = \frac{\frac{Z_0}{\sinh \theta}}{Z_0 \coth \theta} = \frac{1}{\cosh \theta} \quad . \quad . \quad . \quad . \quad (12)$$

If θ is very great (12) becomes

$$\frac{1}{e^{\theta}} = \frac{1}{e^{\gamma l}} = e^{-\gamma l}$$

which agrees with (4) Chapter II.

50. Voltage of a Transmission Line at the Receiving End when the Circuit is open at that End.—By inspection of Fig. 12 the voltage at the end of the circuit B is the same as that at the junction of R_1 with R_2 when the line is disconnected at B. The voltage therefore falls from E_1 to the point where R_1 is earthed, and the drop in R_2 and R_1 is in direct proportion to their resistances. At the junction of R_1 with R_2 the voltage is $E_2 = \frac{E_1 R_1}{R_1 + R_2}$

By (12)
$$\frac{R_1}{R_1 + R_2} = \frac{1}{\cosh \theta}$$

$$\therefore E_2 = \frac{E_1}{\cosh \theta} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

51. Remarks on the Formulæ derived for Finite Lines without Apparatus.—Formulæ (5) to (13) are the usual expressions for the electrical conditions at the ends of circuits having uniformly distributed constants, and they are the same as those proved in Appendix I. for the alternating current case. By the aid of the equivalent circuit diagram the few cases examined have enabled us readily to arrive at the appropriate formulæ expressed in hyperbolic functions, but this point will be better appreciated when the complete chapter has been read. The advantages of expressing formulæ in terms of hyperbolic functions are, however, sufficiently evident at this stage, but the following remarks may make the matter clearer. Although the equivalent circuit is of great use in enabling us to arrive at a correct formula expressed in terms of R_1 and R_2, yet the equivalent expression in hyperbolic functions permits of a wider interpretation than is possible from examination of the result expressed in terms of R_1 and R_2. Take for example $Z_1 = Z_0 \tanh \theta$ (6). From this we are enabled to easily understand the law followed by a uniform circuit of any length having a characteristic resistance Z_0 and a line angle θ. As an example, if a curve be constructed for $\tanh \theta$ between various limits, the way

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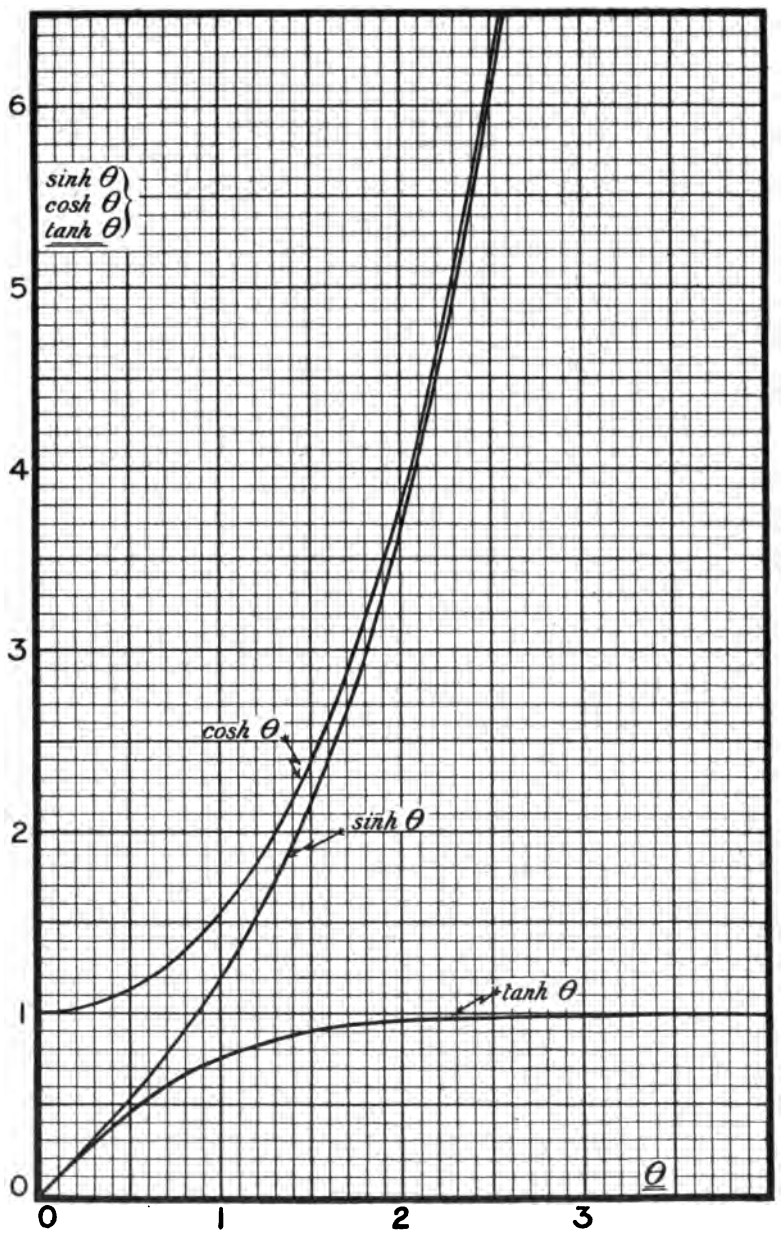

FIG. 14.

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in which the circuit resistance varies between those limits is seen by inspection. Fig. 14 has in fact been constructed to show values of sinh θ, cosh θ and tanh θ. It will be seen from the tanh θ curve that in the case of a line closed at the receiving end (the case shown by formula (6)), when the line angle θ is small the resistance increases rapidly with length, but when the angle θ approaches 2 the resistance changes very slowly with variation of length, and soon becomes practically constant for any greater value of θ. The received current by (10) is inversely proportional to sinh θ, and therefore the variation of current with variation of θ may be deduced from the curve, and so on. These examples show the advantage of expressing transmission formulæ in terms of hyperbolic functions (see also para. 13, Chapter I.), especially as tables are available which reduce the work of calculation to a minimum (see the end of Chapter I. where the names of such tables are given).

52. Measurement of the Electrical Constants of a Transmission Circuit.—From the measurement of the open and closed impedances of a finite circuit the values of Z_0, θ, R, and G may be obtained thus. When the measurements have been made, (5) and (6) are known, and Z_0 is obtained by the method shown in para. 44. Next divide (6) by Z_0 and we obtain tanh θ. From this by reference to a suitable table θ is found. If l is known we then have $\frac{\theta}{l} = \gamma$. Also by (8) Chapter II. $\frac{\gamma}{Z_0} = G$, and by (7) Chapter II. $\gamma Z_0 = R$ (see also Chapter X.).

53. Comparison of the Current at the Receiving End of a Transmission Current of any Length, earthed or short-circuited without Apparatus, with the Current at the Same Point when the Same Line is extended indefinitely.—By (2) Chapter II. the current at any point l in an infinite line is

$$I_2 = \frac{E_1}{Z_0} e^{-\gamma l} \text{ (where } \gamma l = \theta)$$

Also by (10) the current at the receiving end of a transmission line of any length is

$$I'_2 = \frac{E_1}{Z_0 \sinh \theta}, \text{ and } \therefore \frac{I'_2}{I_2} = \frac{1}{e^{-\theta} \sinh \theta} = \frac{1}{e^{-\theta}\left(\frac{e^{\theta} - e^{-\theta}}{2}\right)}$$

that is

$$\frac{I'_2}{I_2} = \frac{2}{1 - e^{-2\theta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The relation between θ and $\frac{I'_2}{I_2}$ is plotted as a curve in Fig. 15. The values of θ are plotted as abscissæ and $\frac{I'_2}{I_2}$ as ordinates, between the limits $\theta = 0{\cdot}2$ and $\theta = 4$. The curve shows that for small values of θ the value of $\frac{I'_2}{I_2}$ is large, *i.e.* there is a great difference between the received current at a given point l on the line if the circuit is

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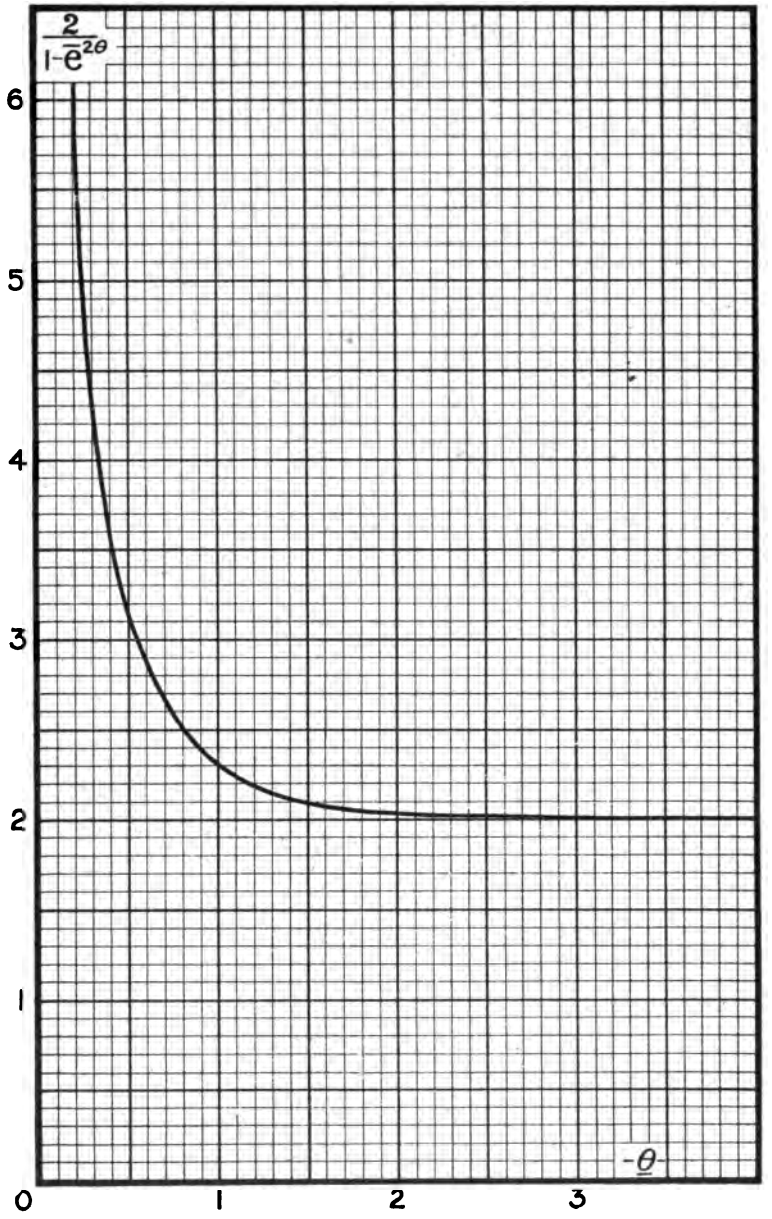

FIG. 15.

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earthed, as compared with the current at the same point if the line is indefinitely extended. When, however, θ approaches 2 the rate of change becomes very small, and for values greater than 2 the ratio $\frac{I'_2}{I_2}$ is practically constant. In the limit when θ is very great calculation shows that $\frac{I'_2}{I_2} = 2$, *i.e.* the current at the receiving end of a long short-circuited line has always twice the value of the current at the same point on the infinite line. If, however, two long finite lines are compared with each other, then since each has a value of current at the receiving end which is twice the magnitude of the current on the infinite line, at the same point, therefore the comparative value of the two currents (*i.e.* their ratio) is the same as if the two lines were extended to infinity.

54. Resistance at the Sending End of a Uniform Transmission Line of any Length, closed by a Resistance Za at the Receiving End.—This and similar problems may be solved by an application of the method already applied in the case of lines without apparatus. In view of the fact that the transmission lines under investigation have generally apparatus at their ends, these problems are of great importance.

The equivalent circuit diagram is given in Fig. 16.

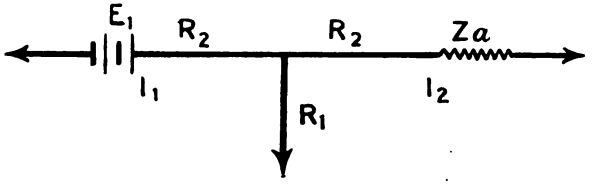

FIG. 16.

The resistance of the equivalent circuit from the junction of R_1 with R_2 is $\frac{R_1(R_2 + Za)}{R_1 + R_2 + Za}$.

The sending end resistance from A is then by inspection of Fig. 16—

$$Z_1 = R_2 + \frac{R_1(R_2 + Za)}{R_1 + R_2 + Za}$$

Rearranging we have

$$Z_1 = \frac{R_1R_2 + R_2^2 + R_2Za + R_1R_2 + R_1Za}{R_1 + R_2 + Za}$$

By substitution from (3) and (4) we have

$$Z_1 = \frac{(R_2^2 + 2R_1R_2) + Za(R_1 + R_2)}{(R_1 + R_2) + Za} = \frac{Z_0^2 + Za(Z_0 \coth \theta)}{Z_0 \coth \theta + Za}$$

Multiply the numerator and denominator by sinh θ and we obtain

$$Z_1 = \frac{Z_0(Z_0 \sinh \theta + Za \cosh \theta)}{Z_0 \cosh \theta + Za \sinh \theta} \quad . \quad . \quad (15)$$

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By (6), which is the case of a line without apparatus, we have $Z_1 = Z_0 \tanh \theta$, and this may be written $Z_0 \frac{(Z_0 \sinh \theta)}{(Z_0 \cosh \theta)}$.

Now (15) is the same expression with $Za \cosh \theta$ added to the numerator and $Za \sinh \theta$ to the denominator. If θ is small $Za \cosh \theta$ is much greater than $Z_0 \sinh \theta$, and it follows that in the case of a line with a small angle θ (generally short lines) the effect of apparatus added at the receiving end will be relatively greater than when θ is great, for in the latter case $\cosh \theta$ and $\sinh \theta$ gradually approach equality (see Fig. 14). In the limit when the line is indefinitely long and therefore θ is great, (15) becomes approximately

$$Z_1 = \frac{Z_0\left(Z_0\frac{e^\theta}{2} + Za\frac{e^\theta}{2}\right)}{\left(Z_0\frac{e^\theta}{2} + Za\frac{e^\theta}{2}\right)} = Z_0$$

Now (6) also becomes equal to Z_0 when θ is very great, so that in this theoretical case the impedance of the receiving apparatus would have no effect on the sending-end impedance.

55. Current at the Sending End of a Finite Transmission Line closed by a Resistance Za at the Receiving End.—Since $I_1 = \frac{E_1}{Z_1}$ we have by (15)

$$I_1 = \frac{E_1}{Z_0} \cdot \frac{Z_0 \cosh \theta + Za \sinh \theta}{Z_0 \sinh \theta + Za \cosh \theta} \quad . \quad . \quad . \quad (16)$$

56. Current at the Receiving End of a Transmission Line closed at that End by a Resistance Za.—The equivalent circuit diagram is the same as Fig. 16. The current will divide at the junction of R_1 with R_2, and the portion passing through Za is

$$I_2 = \frac{I_1 R_1}{R_1 + R_2 + Za}, \text{ where } I_1 = \frac{E_1}{Z_1}$$

Insert the value of Z_1 from para 54 and we obtain

$$I_2 = \frac{E_1(R_1 + R_2 + Za)}{(R_2^2 + 2R_1R_2) + Za(R_1 + R_2)} \times \frac{R_1}{(R_1 + R_2 + Za)}$$

By cancellation of the factor $(R_1 + R_2 + Za)$ and substitution from (2), (3) and (4) we obtain

$$I_2 = \frac{E_1 \frac{Z_0}{\sinh \theta}}{Z_0^2 + Za(Z_0 \coth \theta)}$$

that is

$$I_2 = \frac{E_1}{Z_0 \sinh \theta + Za \cosh \theta} \quad . \quad . \quad . \quad (17)$$

57. Comparative Magnitude of Received Currents in a Long Line due to Variations in the Resistance of the Receiving Apparatus.—If $Za = Z_0$ the expression for the received current in the equivalent circuit and formula (17) become

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$$I_2 = \frac{I_1 R_1}{R_1 + R_2 + Z_0} = \frac{E_1}{Z_0(\sinh\theta + \cosh\theta)} = \frac{E_1 e^{-\theta}}{Z_0}$$

[See (44) Chap. I.]

also $$\frac{I_2}{I_1} = \frac{R_1}{R_1 + R_2 + Z_0} = e^{-\theta} \quad . \quad . \quad . \quad . \quad (18)$$

This is true for any length of line so long as the circuit is closed by a resistance Z_0 (App. II., para. 5). Substitute the value of $R_1 + R_2$ from (4) in this equation and we obtain

$$e^{-\theta} = \frac{R_1}{Z_0 \coth\theta + Z_0}$$

When θ is very large coth θ has the value 1, and we then get $R_1 + R_2 = Z_0$ and

$$e^{-\theta} = \frac{R_1}{2Z_0}$$

Construct the equivalent diagram for this case (Fig. 17). In this case $\frac{Z_0}{\sinh\theta} = R_1$ becomes $2Z_0 e^{-\theta}$.

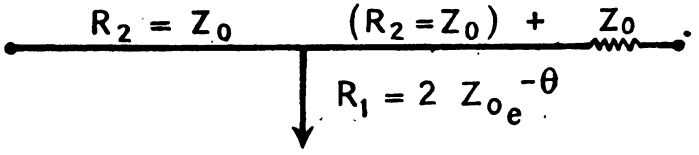

FIG. 17.

It is proved later in this chapter (para. 70) that when θ is very large (say θ=5) R_2 in Fig. 17 is more than 70 times R_1 in magnitude, approximately, therefore, in such a case $Z_0 = R_2$, and this is true whether the long line is closed by an apparatus having a resistance Z_0 or not.

58. To find the Effect on the Received Current at the End of an Indefinitely Long Line, of Two Sets of Receiving Apparatus having Resistances, Z_0 and Za respectively, taken One at a Time.—From the preceding remarks the received currents I_2 and I'_2 in the two cases are

$$(a)\ I_2 = \frac{I_1 R_1}{2Z_0}$$

$$(b)\ I'_2 = \frac{I_1 R_1}{Z_0 + Za}$$

Since $R_2 = Z_0$ in each case the currents at the sending end will be equal, and therefore dividing (*b*) by (*a*), the current I'_2 in Za may be expressed thus—

$$I'_2 = \frac{2I_2 Z_0}{Z_0 + Za} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

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The meaning of this is that if a long line of resistance Z_0 is closed by an instrument having a resistance Za, the current received in Za is $\frac{2Z_0}{Za+Z_0}$ times that received in an instrument having the same resistance as that of the line, and substituted for Za. The same result is arrived at in Chapter VII. by the study of the effects of reflection.

59. Receiving-End Resistance in a Uniform Transmission Line of any Length closed by a Resistance Za at the Receiving End.—The ratio of the impressed volts at the sending end to the received current is called the "receiving-end resistance." We have again, as in (11),

$$\frac{E_1}{I_2} = Z_2$$

whence by (17)

$$Z_2 = (Z_0 \sinh \theta + Za \cosh \theta) \quad . \quad . \quad . \quad (20)$$

This is the required resistance.

60. Sending-End Resistance of a Transmission Line of any Length closed at each End by a Resistance Za.—The equivalent circuit diagram is

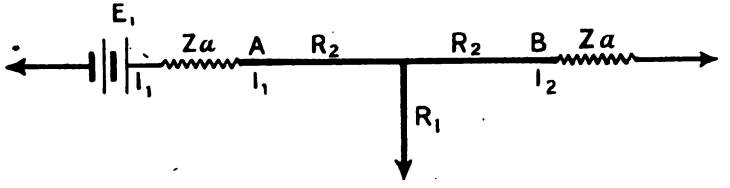

FIG. 18.

The resistance from A (Fig. 18) is the same as (15), and by the addition of Za we have in this case

$$Z_1 = Za + \frac{R_2^2 + 2R_1R_2 + Za(R_1 + R_2)}{R_1 + R_2 + Za}$$

that is $$Z_1 = \frac{ZaR_1 + ZaR_2 + Za^2 + R_2^2 + 2R_1R_2 + Za(R_1 + R_2)}{R_1 + R_2 + Za}$$

Rearranging, and by substitution from (3) and (4), we obtain

$$\begin{aligned} Z_1 &= \frac{Za^2 + (R_2^2 + 2R_1R_2) + 2Za(R_1 + R_2)}{(R_1 + R_2) + Za} \\ &= \frac{Za^2 + Z_0^2 + 2Za(Z_0 \coth \theta)}{Z_0 \coth \theta + Za} \\ &= \frac{(Za^2 + Z_0^2) \sinh \theta + 2ZaZ_0 \cosh \theta}{Z_0 \cosh \theta + Za \sinh \theta} \\ &= \frac{Z_0\left\{\frac{(Za^2 + Z_0^2)}{Z_0} \sinh \theta + 2Za \cosh \theta\right\}}{Z_0 \cosh \theta + Za \sinh \theta} \quad . \quad . \quad (21) \end{aligned}$$

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61. Sending-End Current in a Transmission Line closed by a Resistance Za at both ends of the Line.—As in (16) we have

$$\frac{E_1}{Z_1} = I_1$$

whence
$$I_1 = \frac{E_1}{Z_0}\left\{\frac{Z_0 \cosh\theta + Za \sinh\theta}{\left(\frac{Za^2 + Z_0^2}{Z_0}\right)\sinh\theta + 2Za\cosh\theta}\right\} \quad . \quad (22)$$

If the line is very long this reduces to

$$I_1 = \frac{E_1}{Z_0}\left(\frac{Z_0 + Za}{\frac{Z_0^2 + Za^2}{Z_0} + 2Za}\right) = E_1\frac{Z_0 + Za}{(Z_0 + Za)^2} = \frac{E_1}{Z_0 + Za}$$

62. Receiving-End Current in a Transmission Line closed at each End by Equal Resistances.—The received current is (see Fig. 18)

$$I_2 = I_1\frac{R_1}{R_1 + R_2 + Za}$$

Substituting the value of $I_1 = \frac{E_1}{Z_1}$ from para. 60 we have

$$I_2 = \frac{E_1(R_1 + R_2 + Za)}{Za^2 + (R_2^2 + 2R_1R_2) + 2Za(R_1 + R_2)} \times \frac{R_1}{R_1 + R_2 + Za}$$

By cancelling and substitution from (2), (3) and (4) we obtain

$$I_2 = \frac{E_1R_1}{Za^2 + (R_2^2 + 2R_1R_2) + 2Za(R_1 + R_2)} = \frac{E_1\frac{Z_0}{\sinh\theta}}{Za^2 + Z_0^2 + 2ZaZ_0\coth\theta}$$

$$= \frac{E_1}{\left(\frac{Za^2 + Z_0^2}{Z_0}\right)\sinh\theta + 2Za\cosh\theta} \quad . \quad . \quad (23)$$

If the line is very short $\sinh\theta$ is very small and $\cosh\theta$ approaches 1. In the limit when $\theta = 0$ we have

$$I_2 = \frac{E_1}{2Za}$$

which is obvious, since there is then nothing in circuit but the apparatus $2Za$ and the battery E_1.

If the line is very long both $\sinh\theta$ and $\cosh\theta$ approximate to $\frac{e^\theta}{2}$ and we obtain

$$I_2 = \frac{E_1}{\frac{Za^2 + Z_0^2}{Z_0}\cdot\frac{e^\theta}{2} + 2Za\frac{e^\theta}{2}}$$

$$= \frac{E_1}{\frac{e^\theta}{2}\left(\frac{Za^2 + Z_0^2 + 2ZaZ_0}{Z_0}\right)}$$

$$= \frac{E_1e^{-\theta}(2Z_0)}{(Za + Z_0)^2} = \frac{E_1e^{-\theta}}{2Za} \times \frac{4Z_0Za}{(Za + Z_0)^2} \quad (24)$$

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If $Za = Z_0$ this becomes

$$I_2 = \frac{E_1 e^{-\theta}}{2Z_0} \quad \text{or} \quad \frac{E_1}{I_2} = 2Z_0 e^{\theta} \quad . \quad . \quad . \quad (25)$$

If, however, $Za = Z_0$ in (23), we also obtain

$$I_2 = \frac{E_1 e^{-\theta}}{2Z_0} .$$

Now (23) is true for a line of any length, whereas (25) is merely derived from the equation for a long line. The reason for the identical result in the two cases is simple: when a line is closed by a resistance equal to its characteristic resistance it acts as an infinite line, no matter how long it may be. In such a case $Z_0 + Za = 2Z_0$ as shown in (25); this is the sending-end impedance, as may be verified by putting $Z_0 = Za$ in (21). The resistance of the receiving apparatus does not now appear in the equation. (See the remarks on the infinite line in Appendix II., 5.)

63. Receiving-End Resistance of a Transmission Line of any Length closed at each End by a Resistance Za.—The process is the same as in previous cases. We have

$$\frac{E_1}{I_2} = Z_2$$

that is

$$Z_2 = \left\{ \frac{(Za^2 + Z_0^2)}{Z_0} \sinh \theta + 2Za \cosh \theta \right\} . \quad . \quad (26)$$

If $Z_0 = Za$ this becomes $Z_2 = 2Za\ (\sinh \theta + \cosh \theta) = 2Zae^{\theta}$, or $2Z_0 e^{\theta}$.

64. Sending-End Resistance of a Transmission Line of any Length closed at the Sending End by a Resistance Za**, and at the Receiving End by a Resistance** Zb.—The equivalent circuit diagram is shown in Fig. 19.

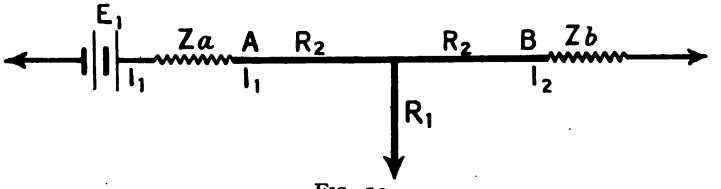

FIG. 19.

It is evident from the method adopted in dealing with the preceding examples that the resistance from A is

$$Z_1 = Za + R_2 + \frac{R_1(R_2 + Zb)}{R_1 + R_2 + Zb}$$

Rearranging and by substitution from (3) and (4) we obtain

$$Z_1 = \frac{ZaZb + (R_2^2 + 2R_1R_2) + (Za + Zb)(R_1 + R_2)}{(R_1 + R_2) + Zb}$$

$$= \frac{ZaZb + Z_0^2 + (Za + Zb)Z_0 \coth \theta}{Z_0 \coth \theta + Zb} \quad . \quad . \quad . \quad . \quad (27)$$

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Multiplying numerator and denominator by sinh θ and simplifying we have

$$Z_1 = \frac{Z_0\left\{\left(\frac{ZaZb + Z_0^2}{Z_0}\right)\sinh\theta + (Za + Zb)\cosh\theta\right\}}{Z_0\cosh\theta + Zb\sinh\theta} \quad (28)$$

65. Receiving-End Current in a Transmission Line closed at One End by a Resistance Za and at the Receiving End by a Resistance Zb.—The received current is by inspection of Fig. 19

$$I_2 = \frac{I_1 R_1}{R_1 + R_2 + Zb}$$

where $I_1 = \frac{E_1}{Z_1}$

Substitute Z_1 from the left-hand side of (27) and we obtain

$$I_2 = \frac{E_1(R_1+R_2+Zb)}{(ZaZb)+(R_2^2+2R_1R_2)+(Za+Zb)(R_1+R_2)} \times \frac{R_1}{R_1+R_2+Zb} \quad (29)$$

By cancelling and substitution from (2), (3) and (4) we obtain

$$I_2 = \frac{E_1 \frac{Z_0}{\sinh\theta}}{(ZaZb) + Z_0^2 + (Za + Zb)Z_0\coth\theta}$$

$$= \frac{E_1}{\frac{(ZaZb + Z_0^2)\sinh\theta}{Z_0} + (Za + Zb)\cosh\theta} \quad (30)$$

66. Receiving-End Resistance of a Transmission Line closed at One End by a Resistance Za, and at the Other End by a Resistance Zb.—We have $\frac{E_1}{I_2} = Z_2$, whence by (30)

$$Z_2 = \left\{\frac{(ZaZb + Z_0^2)\sinh\theta}{Z_0} + (Za + Zb)\cosh\theta\right\} . \quad (31)$$

67. To find the Sending-End Current of a Transmission Line closed at One End by a Resistance Za, and at the Other End by a Resistance Zb.—We have $\frac{E_1}{I_1} = Z_1$, whence by (28)

$$I_1 = E_1\left[\frac{Z_0\cosh\theta + Zb\sinh\theta}{Z_0\left\{\left(\frac{ZaZb + Z_0^2}{Z_0}\right)\sinh\theta + (Za + Zb)\cosh\theta\right\}}\right] \quad (32)$$

68. To show that the Received Current is the Same in the Circuit depicted in Fig. 19 whether the Voltage E_1 is applied at the End Za or Zb.—The current I_2 with the voltage applied at the end Za has already been found and the result is given in (29) and (30).

If the voltage be applied at the end Zb the resistance measured from that point is by inspection of Fig. 19

$$Z_1 = Zb + R_2 + \frac{R_1(R_2 + Za)}{R_1 + R_2 + Za}$$

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Rearranging we obtain

$$Z_1 = \frac{ZaZb + (R_2^2 + 2R_1R_2) + (Za + Zb)(R_1 + R_2)}{(R_1 + R_2) + Za}$$

This is the sending-end resistance from Zb, but it is not the same as (27), which is the sending-end resistance from Za.

The received current is, however, by inspection of Fig. 19,

$$I_2 = \frac{I_1R_1}{R_1 + R_2 + Za}$$

Substitute the value of the current from $I_1 = \frac{E_1}{Z_1}$. We then have

$$I_2 = \frac{E_1(R_1 + R_2 + Za)}{ZaZb + (R_2^2 + 2R_1R_2) + (Za + Zb)(R_1 + R_2)} \times \frac{R_1}{R_1 + R_2 + Za}$$

$$= \frac{E_1R_1}{ZaZb + (R_2^2 + 2R_1R_2) + (Za + Zb)(R_1 + R_2)}$$

This is the same as (29), which proves that in such a case equal currents are received at the distant end of the line when the same voltage is applied at either end irrespective of the relative resistance of the apparatus at the ends.

The resistance at either end of the line may have any value whatever. Assuming Zb to be 0, it will be seen that the received current is the same with the same applied voltage at either end of the line. The condition referred to is shown by Fig. 16.

The same proposition may be proved for other combinations of lines and apparatus in series (see Chapter XIII.).

69. The Equivalence of the T and Π Circuits and the relation of their Constants.—In Appendixes III. and IV. it is proved that either a T or Π circuit may be constructed, which is equivalent for operations from the two ends to a real circuit. It will now be shown that when the constants of either the T or the Π circuit have been found the constants of the other circuit may easily be deduced from them. Figs. 9 and 11 show the equivalent T circuit, and Figs. 20 and 21 the equivalent Π circuit.

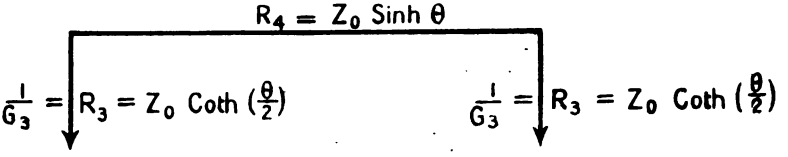

FIG. 20.

If Figs. 9 and 20 represent the same uniform line, we have

$$R_2R_3 = Z_0 \tanh \frac{\theta}{2} Z_0 \coth \frac{\theta}{2} = Z_0^2$$

(from Appendixes III. and IV.), whence

$$R_3 = \frac{Z_0^2}{R_2} \quad \ldots \ldots \quad (33)$$

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Also $R_1R_4 = \frac{Z_0}{\sinh \theta} Z_0 \sinh \theta = Z_0^2$

(see (5) App. III., and (21) App. IV.), whence

$$R_4 = \frac{Z_0^2}{R_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Since R_2R_3 and R_1R_4 each equal Z_0^2

$$\therefore R_2R_3 = R_1R_4 \text{ and } \frac{R_1}{R_2} = \frac{R_3}{R_4} \quad . \quad . \quad . \quad (35)$$

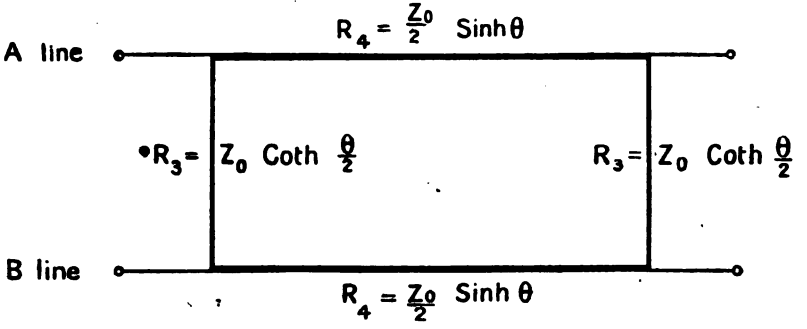

FIG. 21.

It can also be shown that

$$R_2 = Z_0 \tanh \frac{\theta}{2} = Z_0 \frac{\cosh \theta - 1}{\sinh \theta} = R_1(\cosh \theta - 1).$$

(See Chap. I. (54), and (5) App. III.)

Divide by $R_1 = \frac{Z_0}{\sinh \theta}$ and we obtain

$$\frac{R_2}{R_1} = \frac{R_4}{R_3} = \frac{Z_0(\cosh \theta - 1)}{\sinh \theta} \times \frac{\sinh \theta}{Z_0} = \cosh \theta - 1 \quad (36)$$

70. Deduction from Equation (36).—When θ approaches 0 the value $\cosh \theta - 1$ is extremely small, and in the limit when $\theta = 0$ $\cosh \theta - 1 = 0$, that is $\frac{R_2}{R_1} = 0$, or in other words, R_2 is incomparably small in comparison with R_1. As θ increases, the value of $\cosh \theta - 1$ increases until $\theta = 1{\cdot}317$ when $\cosh \theta - 1 = 1$, and therefore $R_2 = R_1$. For greater values of θ, $\cosh \theta - 1$ is greater than 1, and continually increases; when $\theta = 5$, for example, $\cosh \theta - 1 = 74{\cdot}2$.

It is evident from the preceding remarks and examples that R_2, which represents the value of half the equivalent T circuit, continually increases in ratio with respect to $R_1 \left(\text{or } \frac{1}{G}\right)$ with increasing values of θ, until in a very long circuit (*i.e.* with a large value of θ) such as may exist in practice, R_2 is more than 70 times R_1. Applying these remarks to the case of a real circuit with uniformly distributed leakance and resistance, it will be evident that when the line is short the conductor resistance is small and the insulation

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resistance comparatively high; as the circuit increases in length the resistance of the conductor continually increases and the insulation resistance decreases, the combined effect in a long circuit is that the sending-end resistance becomes constant. (See Appendix I.)

EXAMPLES ON CHAPTER III.

No. 1. A single wire Cu conductor having a resistance of 25·14 ohms per mile and a leakance of 1 micro-mho per mile, has a line angle $\theta=1$. Having given that $Z_0=\sqrt{\frac{R}{G}}$ show that in the D.C. case—

(*a*) $Z_0=\sqrt{R_2(2R_1+R_2)}$

(*b*) $Z_1=\frac{R_2(2R_1+R_2)}{R_1+R_2}=Z_0\tanh\theta$. [See formula (6).]

Now R = 25·14 ohms per mile
and G = 1 × 10^{-6} mhos per mile

$$\therefore\ Z_0=\sqrt{\frac{25\cdot14}{10^{-6}}}=5\cdot014\times10^3=5014 \text{ ohms}$$

(*a*) In the equivalent circuit for this line (see Fig. 13), and using formulæ (1) and (2)

$$R_2=Z_0\tanh\frac{\theta}{2}=5014\times\tanh 0\cdot5=2317 \text{ ohms}$$

$$R_1=\frac{Z_0}{\sinh\theta}=\frac{5014}{\sinh 1}=4266\cdot5 \text{ ohms}$$

also $\sqrt{R_2(2R_1+R_2)}=\sqrt{2317\times10850}=5014$ ohms $=Z_0$
[See formula (3).]

(*b*) $Z_0\tanh\theta=5014\times\tanh 1=3819$ ohms

$$\frac{R_2(2R_1+R_2)}{R_1+R_2}=\frac{10850\times2317}{6583\cdot5}=3819 \text{ ohms}$$

By verifying in this way that equations expressed in terms of ordinary resistance give the same result as hyperbolic functions, familiarity with, and confidence in the latter will be obtained.

No. 2. For a circuit having the same constants as that given in Example No. 1, construct the equivalent T and Π circuits.
See Figs. 9 and 20. For proofs see Appendixes III. and IV.

The values of R_1 and R_2 have been calculated in the previous example, hence the equivalent T circuit is

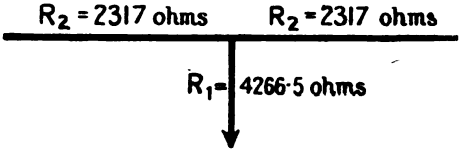

FIG. 22.

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From (33) $R_3 = \frac{Z_0^2}{R_2} = \frac{5014^2}{2317} = 10850$ ohms

From (34) $R_4 = \frac{Z_0^2}{R_1} = \frac{5014^2}{4266{\cdot}5} = 5892$ ohms

The equivalent II circuit is shown in Fig. 23.

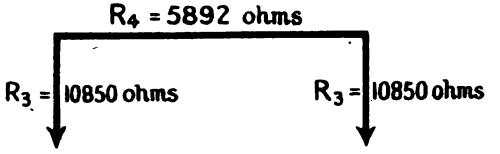

FIG. 23.

No. 3. A Cu conductor weighing 150 lbs. per ml. has an insulation resistance of 0·5 megohm per mile, and a line angle $\theta = 1$.

(*a*) Find the nominal insulation and resistance of the line (sometimes called the apparent insulation and resistance) if the total leakance (Gl) is lumped in the middle of the total resistance (Rl), and

(*b*) Find the insulation and the conductor resistance (as measured at the sending end) if formulæ (5) and (6) are used to determine the results.

(*a*) Now $\gamma = \sqrt{RG} = \sqrt{\frac{11{\cdot}73}{2} \times 2 \times 10^{-6}} = {\cdot}003425$ hyp. per ml.

and $\gamma l = 1$

$$\therefore l = \frac{1}{{\cdot}003425} = 291{\cdot}98 \text{ mls.} \quad [\text{See (5), Chap. II.}]$$

$$Rl = \frac{11{\cdot}73 \times 291{\cdot}98}{2} = 1712 \text{ ohms.} \quad \text{Call this value } x.$$

$$\frac{1}{Gl} = \frac{0{\cdot}5 \times 10^6}{291{\cdot}98} = 1712 \text{ ohms.} \quad \text{,, ,, } y,$$

Therefore Nominal Insulation Resistance (see Fig. 24, and Fig. 10 with the remarks referring to it)

$$= \frac{x}{2} + y = 856 + 1712 = 2568 \text{ ohms}$$

and Nominal Line Resistance (see Fig. 25)

$$= \frac{x}{2} + \frac{\frac{xy}{2}}{\frac{x}{2} + y} = 856 + \frac{856 \times 1712}{2568} = 1427 \text{ ohms}$$

(*b*) Insulation Resistance as measured at the sending end of the real line (see Fig. 26)

$$= Z_0 \coth \theta \text{ [from (5)]}$$
$$= 1712 \times 1{\cdot}3130 = 2248 \text{ ohms}$$

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Line Resistance as measured at the sending end of the real line (see Fig. 27)

$$= Z_0 \tanh \theta \text{ [from (6)]}$$
$$= 1712 \times \cdot 7616 = 1304 \text{ ohms}$$

To construct the equivalent circuit R_1 and R_2 must be calculated.

Now $$R_1 = \frac{Z_0}{\sinh \theta} = \frac{1712}{1 \cdot 1752} = 1457 \text{ ohms}$$

and $$R_2 = Z_0 \tanh \frac{\theta}{2} = 1712 \times \cdot 4621 = 791 \text{ ohms}$$

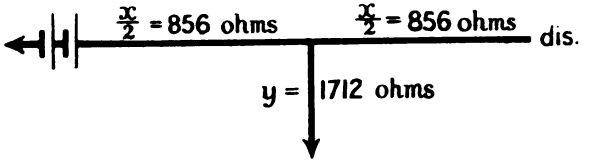

FIG. 24.

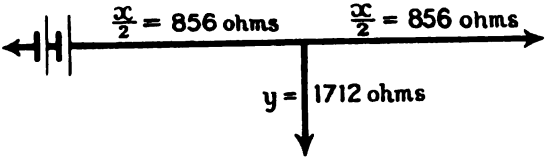

FIG. 25.

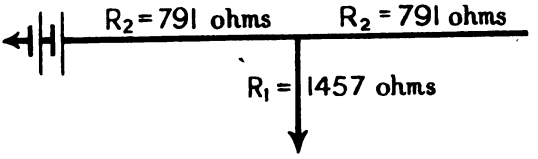

FIG. 26.

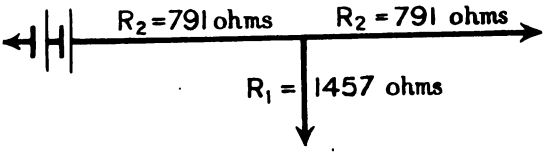

FIG. 27.

No. 4. (*a*) A single wire Cu conductor weighing 150 lbs. per ml. has a line angle $\theta = 1$ and a leakance of 2×10^{-6} mhos per mile. It is earthed direct at the receiving end. Find the length of the line; also the received current if 10 volts are applied to the sending end.

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(*b*) At each end of this line similar apparatus of 400 ohms resistance is joined in series: 10 volts are applied at the sending end and the receiving end earthed. Find the received current.

(*c*) What is the length of the line without apparatus if the received current is the same as found in (*b*), when 10 volts are applied at the sending end and the receiving end is earthed?

(*a*) From Example No. 3 (*a*), the length of the line is 291·98 mls.

$$Z_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{11{\cdot}73}{2 \times 2 \times 10^{-6}}} = 1712{\cdot}4 \text{ ohms}$$

$$I_2 = \frac{E_1}{Z_0 \sinh \theta} \text{ [from (10)]}$$

$$= \frac{10}{1712{\cdot}4 \times \sinh 1}$$

$$= \frac{10}{1712{\cdot}4 \times 1{\cdot}1752} = {\cdot}004969 \text{ amps.}$$

(*b*) $$I_2 = \frac{E_1}{\left(\frac{Z_0^2 + Za^2}{Z_0}\right) \sinh \theta + 2Za \cosh \theta} \text{ [from (23)]}$$

$$= \frac{10}{\left(\frac{1712{\cdot}4^2 + 400^2}{1712{\cdot}4}\right) \times 1{\cdot}1752 + 800 \times 1{\cdot}54308}$$

$$= \frac{10}{1805{\cdot}8 \times 1{\cdot}1752 + 1234{\cdot}5}$$

$$= \frac{10}{2122{\cdot}2 + 1234{\cdot}5}$$

$$= \frac{10}{3356{\cdot}7} = {\cdot}002979 \text{ amps.}$$

(*c*) Now $$I_2 = \frac{E_1}{Z_0 \sinh \theta} \text{ [from (10)]}$$

$$\therefore \sinh \theta = \frac{E_1}{I_2 Z_0} = \frac{10}{{\cdot}002979 \times 1712{\cdot}4} = 1{\cdot}9603$$

hence $$\theta = 1{\cdot}426 .$$

but $\theta = \gamma l$

$$\therefore l = \frac{\theta}{\gamma} = \frac{1{\cdot}426}{{\cdot}003425} = 416{\cdot}4 \text{ mls.}$$

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CHAPTER IV

THE LOADING OF TRANSMISSION LINES AND THE DESIGN OF ARTIFICIAL CABLES—DIRECT CURRENT CASE

71. Introductory Remarks.—The method here adopted of finding the effect of inserting lumped loads in a uniform transmission line in series or leak is as follows. In the first instance, a T or Π circuit which is equivalent to the unloaded uniform transmission line is calculated. Loads are next added to the equivalent T or Π circuit, and the formulæ for the modified equivalent loaded circuit is developed. This loaded equivalent T or Π circuit is equivalent to some uniform transmission line, having electrical constants R, L, G, and C, which can be found. If the uniform transmission line thus found is compared with the uniform unloaded transmission line the effect of loading may be seen. Dr. Kennelly has dealt extensively with this method in the book referred to in para. 41.

Consider an equivalent T circuit such as that shown in Fig. 9. As already shown, this circuit can be made equivalent in the direct current case for all operations conducted from its two ends, to a uniform transmission line having a characteristic resistance Z_0 and a line angle θ. It should be noticed that the proof of the T circuit in Appendix III. stipulates that the leak resistance R_1 is independent of the conductor resistance R_2. This is evident if the physical circuit be considered. Take, for example, a conductor on a line of poles where the insulators have a given combined resistance per unit length of the line. If the resistance of the conductor is changed, and nothing else, we may assume the resistance of the leak to remain unchanged, if it is independent of the applied voltage. Suppose that the resistance of the conductor R_2 changes whilst R_1, the equivalent resistance of the leak, and also the circuit length, remain unchanged.

In that case the values of Z_0 and θ will necessarily change, seeing that the value of each depends partly on the resistance of the line per unit length. It follows that if $R_1 = \frac{Z_0}{\sinh \theta}$ before the alteration it will have a value $R_1 = \frac{Z'_0}{\sinh \theta_1}$ after the alteration, where Z'_0 and θ_1 are the new values of the characteristic resistance and the line angle, brought about by the change in the conductor resistance. It also follows that $Z_0 \sinh \theta_1 = Z'_0 \sinh \theta$, and therefore

$$\frac{Z_0}{Z'_0} = \frac{\sinh \theta}{\sinh \theta_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

since by supposition R_1 is the same in both cases. The same

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reasoning evidently applies to the Π circuit, seeing that there is a fixed relation between the formulæ for the T and the Π circuit by (36), Chap. III.

72. Regular Series Loading of Uniform Lines.—If equal resistances are inserted at regular distances in series with the conductors of a uniform transmission line, it is said to be regularly loaded, and this type of loading is called series loading, to distinguish it from leak loading, in which the resistances are joined in parallel with the conductors. The resistances, which are inserted as described, are called " series loads " or " leak loads," as the case may be.

It may be premised that a reduction in the real part of the line angle θ may be obtained by the introduction of suitable loads when alternating currents of telephonic frequency represent the working conditions of the circuit to be loaded, but the line angle θ as a whole is always increased by loading, and in the case where direct currents are used, there is a consequent increase in the attenuation constant of a loaded transmission line. Loading is no exception to the rule that the same general considerations apply both to alternating and direct current transmission, and as before, on account of the relative simplicity of the direct current case, it will now be investigated.

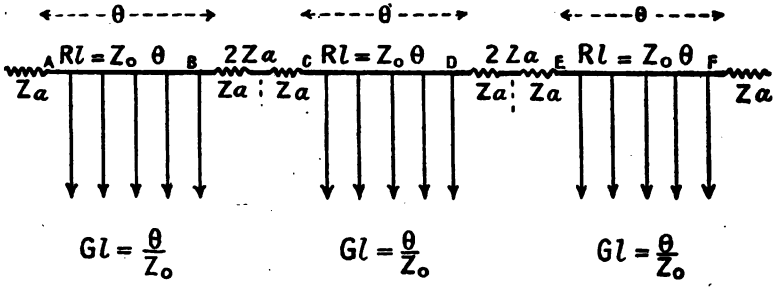

FIG. 28.

Let AF, Fig. 28, represent three uniform sections of line, AB, CD, and EF, loaded with loads of $2Za$ at equal intervals. Each section of loaded conductor for purposes of calculation, may be said to comprise a length of line, such as AB, and the half load Za at each of its ends. Each of the unloaded circuits may be represented by its equivalent T circuit, as in Fig. 9, and each section of uniform line, such as AB with its two end loads Za, offers a similar case to Fig. 18, and the resistance and current at the sending and receiving ends might be directly found by formulæ (21) to (26), Chap. III. If three equivalent T circuits, each of which is equivalent to one of the unloaded lengths, such as AB, Fig. 28, be loaded with a load Za at each end, and then joined in series, we have the arrangement shown in Fig. 29.

In order to find the effect of loading, it is convenient to find the constants of a uniform circuit which is equivalent to one of the loaded sections in Fig. 29, and this circuit, by comparison with the unloaded lengths in Fig. 28, will show the result of adding the

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loads to that section. The constants of the required uniform circuit may be found in various ways, two of which are as follows :—

(*a*) *First Method.*—By formula (23), Chap. III., the received current I_2 at the earthed end of one of the loaded T sections in Fig. 29 may be found (*cp.* Fig. 18). Having thus found I_2, the next step is to find, by the aid of (10), Chap. III., a uniform circuit of the same length as the uniform line AB in Fig. 28, but having characteristics Z'_0 and θ_1 corresponding to the loaded circuit instead of Z_0 and θ as assumed in Fig. 28. By equating (23) to (10) Z'_0 and θ_1 are found. We have

$$\frac{E_1}{\frac{Za^2 + Z_0^2}{Z_0} \sinh \theta + 2Za \cosh \theta} = \frac{E_1}{Z'_0 \sinh \theta_1} \quad . \quad (2)$$

Here the expression on the left gives the current I_2 of the loaded and earthed conductor. Za is the impedance of one load, and Z_0 the characteristic impedance of the line whose angle is θ. All

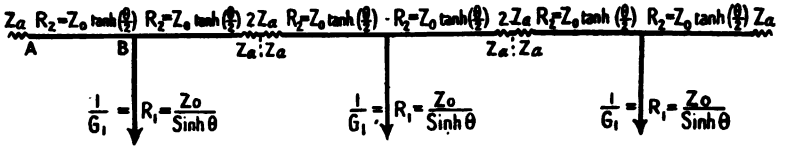

FIG. 29.

these values are known, and therefore I_2 is known. The expression on the right is the formula for the received current in an earthed uniform conductor; E_1 is the same in each case.

From (2) $Z'_0 \sinh \theta_1$ is found, and to find Z'_0 and $\sinh \theta_1$ separately we proceed as follows. The total resistance of the leak R_1 in Fig. 29 is known, and it is unaltered, we therefore have by (1) $\frac{Z_0}{\sinh \theta} = \frac{Z'_0}{\sinh \theta_1}$, and since $Z'_0 \sinh \theta_1$ is already found, we have $\sqrt{\frac{Z'_0}{\sinh \theta_1}} \times Z'_0 \sinh \theta_1 = Z'_0$, and by elimination of Z'_0 from $Z'_0 \sinh \theta_1$, the value of θ_1 is found from a suitable table. The values of R and G are then found from (7) and (8), Chap. II., for $\gamma_1 \times Z'_0 = R$, whence $\gamma_1 l Z'_0 = \theta_1 Z'_0 = Rl$ and similarly $\frac{\theta_1}{Z'_0} = Gl$.

Rl is the total resistance, and Gl the total leakance of a uniform line of length l. The values of Z'_0, θ_1, R, and G have therefore been found, and this completely determines the required uniform circuit, which, by comparison with the unloaded sections of Fig. 28, will show the result of loading the conductors. The sending-end impedance of the circuit given in Fig. 18 is obtained from (21), Chap. III. (see also (30), Chap. XIII.).

(*b*) *Second Method.*—To find the uniform circuit which is equivalent to one of the loaded sections in Fig. 28 construct a rearranged equivalent T diagram of one of the loaded sections of Fig. 29. The resistance R_2, which is half the resistance of one of the unloaded T circuits in Fig. 29, may be regarded as increased

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to the value $R_2+Za=R'_2$, and this increased value may constitute the arm of a new equivalent T circuit; the leak resistance R_1 of this new T circuit being the same as before. The rearranged diagram is shown in Fig. 30.

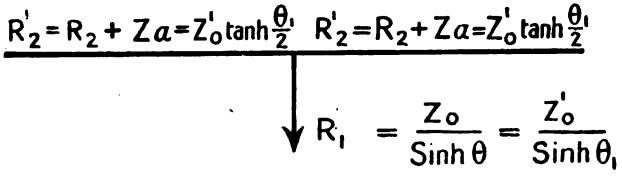

FIG. 30,

By (36), Chap. III., we have $\frac{R'_2}{R_1}=\cosh\theta_1-1$ or $\frac{R'_2}{R_1}+1=\cosh\theta_1$, and from this θ_1 may be found by means of a suitable table of hyperbolic functions. To find Z'_0 we have $R'_2=Z'_0 \tanh \frac{\theta_1}{2}$, from (1), Chap. III., and as R'_2 is known from Fig. 30, and θ_1 has been found, Z'_0 may be found by the aid of a table of hyperbolic functions. Having found Z'_0 and θ_1, the constants R and G of the required uniform line may be found as before. This uniform line, by comparison with the unloaded sections of Fig. 28, shows the effect of loading these conductors.

73. Regular Leak Loading of Conductors. Special Case.—In view of the fact that any equivalent T circuit is related to a corresponding Π circuit, as shown in Chap. III. (33) to (36), it is always possible to find a Π circuit having the same line angle θ as a given T circuit. If this reasoning be applied to Fig. 30, it will be seen that a loaded Π circuit having the same line angle as this loaded T circuit may be found, and this method offers a convenient means of solving the problem. The procedure may be shown as follows. By (36), Chap. III., we have $\frac{R_2}{R_1}=\frac{R_4}{R_3}$. Let R_1 and R_2 be the equivalent resistances of one of the unloaded T circuits, as shown in Fig. 29, then R_3 and R_4 are the resistances of a Π circuit, which is equal to one of the unloaded T circuits in Fig. 29. If the T circuit be loaded, the loaded T circuit will be

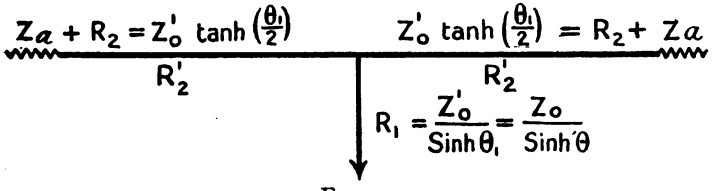

FIG. 31.

that shown in Fig. 30, and its equivalent resistances are R_1 and R'_2, where R_1 is unaltered, *i.e.* it is the same in Figs. 29 and 30. This loaded T circuit may also be represented for our immediate purpose as in Fig. 31.

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The corresponding unloaded Π circuit, as already shown, has equivalent resistances R_3 and R_4. Fig. 20 will show the construction of the diagram. If this circuit be leak loaded so as to obtain the same line angle θ_1 as in the case of the loaded T circuit (Fig. 31), only R_3 need be altered, *i.e.* that portion representing the resistance R_4 will remain unchanged, we have therefore by (36) Chap. III., $\frac{R'_2}{R_1} = \frac{R_4}{R'_3}$, where the left-hand values refer to the loaded T circuit, and the right-hand ones refer to the loaded Π circuit. The diagram of this circuit is given in Fig. 32.

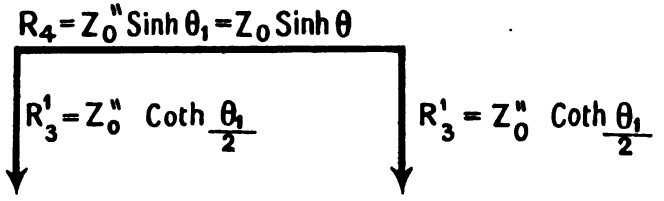

FIG. 32.

When the equivalent resistance R_3' has been found, as explained from the known values of R_2', R_1, and R_4, the value of the leak Ga to be added to $\frac{Gl}{2}$, which is the leakance per mile multiplied by half the length, of the unloaded circuit, is as follows:—

$$\frac{1}{R_3'} - \frac{1}{R_3} = Ga$$

and therefore

$$\frac{1}{R_3'} = \frac{1}{R_3} + Ga = \frac{Gl}{2} + Ga \quad . \quad . \quad . \quad . \quad (3)$$

The value of R_3', expressed in terms of hyperbolic functions, is (App. III. and IV.) by substitution in the equation $\frac{R_1}{R_2'} = \frac{R_3'}{R_4}$

$$\frac{\frac{Z_0}{\sinh \theta}}{Z_0' \tanh \frac{\theta_1}{2}} = \frac{R_3'}{Z_0 \sinh \theta}; \text{ that is } Z_0^2 = Z_0' \tanh \frac{\theta_1}{2} R_3'$$

If we write Z_0'' for $\frac{Z_0^2}{Z_0'}$, we obtain [see (5)]

$$Z_0'' \coth \frac{\theta_1}{2} = R'_3 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

By the foregoing procedure a leak-loaded circuit having the same line angle as the loaded T circuit in Fig. 31 is found, but the characteristic resistances in the two cases are not the same; they are, however, related as follows:

By (33) and (34), Chapter III., $R_1R_4 = Z_0^2 = R_3'R_2'$. Substituting the values of R_2' and R_3', as shown in Figs. 31 and 32, we have

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$$Z_0^2 = R_2'R_3' = Z_0' \tanh \frac{\theta_1}{2} Z_0'' \coth \frac{\theta_1}{2}; \text{ whence } Z_0^2 = Z_0'Z_0''$$

or
$$Z_0 = \sqrt{Z_0'Z_0''} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Any number of circuits similar to Fig. 32 may be placed in series, and as each is equivalent to a section of uniform line, the sum of the Π circuits will be equivalent from the combined circuit ends for all electrical operations conducted from those ends, to any similar operations conducted from the ends of the sum of the sections of the corresponding uniform circuits. This equality also holds for the ends of all the intermediate Π circuits compared with the corresponding point on the uniform circuit. These remarks also apply to T circuits, for example, those shown in Figs. 28 and 29, or any similar combination. If, for example, the end of an intermediate T circuit coincides with the middle of an artificial circuit made up of a number of T circuits, the current and voltage relations at that point are precisely the same as at the middle of the real line.

74. Relative Value of the Series Load and the Leak Load which produce the Same Line Angle θ in a Given Uniform Circuit.—We have already shown how to find the leak load which gives the same line angle as the series loaded circuit. It will now be shown that the series load divided by the leak load $= Z_0^2$ in such a case.

If a series load Za is added to a uniform circuit at intervals of l miles, the resistance $Z'a$ of the load per mile, which is to be added to the circuit resistance per mile, is $Z'a = \frac{Za}{l}$. The total resistance per mile is $R + Z'a$, and therefore the characteristic resistance Z_0' is

$$Z_0' = \sqrt{\frac{R + Z'a}{G}}$$

If a leak load Ga is added to a uniform circuit at intervals of l_1 miles the leakance of the load per mile, which is to be added to the normal leakance G per mile, is $G'a = \frac{Ga}{l_1}$, and the total leakance per mile is $G + G'a$, and therefore the characteristic resistance for the leak-loaded circuit is

$$Z_0'' = \sqrt{\frac{R}{G + G'a}}$$

Multiplying Z_0' by Z_0'' we obtain

$$Z_0'Z_0'' = \sqrt{\frac{R + Z'a}{G}} \times \sqrt{\frac{R}{G + G'a}}$$

$$= \sqrt{\frac{R + Z'a}{G + G'a}} \sqrt{\frac{R}{G}} = \sqrt{\frac{R + Z'a}{G + G'a}} Z_0 \quad (6)$$

where Z_0 is the characteristic resistance of the uniform line. Now since $Z_0'Z_0'' = Z_0^2$ by (5), therefore from (6)

$$Z_0^2 = \sqrt{\frac{R + Z'a}{G + G'a}} Z_0$$

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whence $$Z_0 = \sqrt{\frac{R + Z'a}{G + G'a}} = \sqrt{\frac{R}{G}} \quad . \quad . \quad . \quad (7)$$

by (6), Chap. II., and consequently

$$\sqrt{\frac{Z'a}{G'a}} = \sqrt{\frac{R}{G}} = Z_0$$

The required relative values are therefore $Z'a$ and $G'a$ as given in (7), *et seq.*

75. Regular Leak Loading—General Case.—The special case of leak loading just considered only covers one particular aspect of the case, the problem will now be dealt with from a more general point of view. The mode of procedure is similar to that adopted in the case of series loading dealt with in Figs. 28 and 29. Let Fig. 33 represent a uniform line split up into three sections, AB, CD, and EF, of equal length. The total conductor resistance in each section is $Rl = Z_0\theta$, and the total leakance of each section is $Gl = \frac{\theta}{Z_0}$. One half of this leakance or $\frac{Gl}{2}$ is shown at each end of the uniform line.

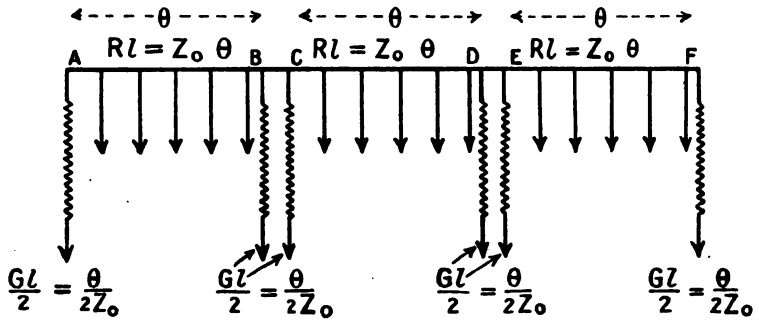

FIG. 33.

In order to leak load such a circuit, leaks Ga are added to each of the normal leaks $\frac{Gl}{2}$, as shown in Fig. 34.

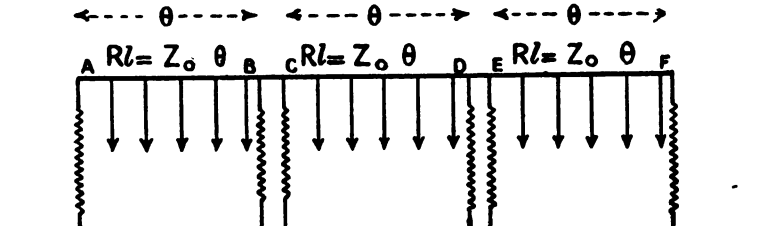

FIG. 34.

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Each intermediate loading point may be regarded as having a leak $Gl+2Ga$ split up into two equal parts, the leak at each end of a section being $\frac{Gl}{2}+Ga$.

Next construct the equivalent circuit diagram of one of the three equal unloaded uniform lengths in Fig. 33. The corresponding Π circuit, given in Fig. 20, shows the construction of the diagram; the equivalent resistances are R_3 and R_4. If this circuit be leak loaded by adding leaks Ga to the leak whose resistance is R_3 (Fig. 20), we obtain a new leak having a resistance such that $\frac{1}{R_3'}-\frac{1}{R_3}=Ga$. In this loading operation the line resistance is not affected, and therefore the value R_4 remains unaltered. Fig. 35 shows a smoothed section of a leak-loaded circuit which is equal to one of the loaded sections shown in Fig. 34.

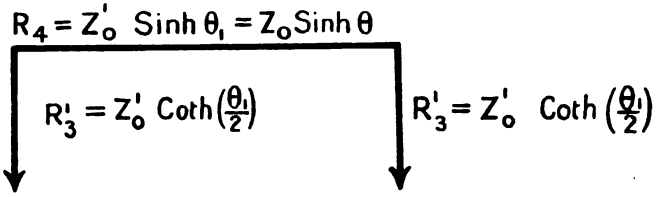

FIG. 35.

The values Z_0' and θ_1 for this circuit may be found as follows. We have from (36), Chapter III., $\frac{R_4}{R_3'}=\cosh\theta_1-1$, or $\frac{R_4}{R_3'}+1=\cosh\theta_1$. From this equation θ_1 may be deduced. To find Z_0' we have $R_3'=Z_0'\coth\frac{\theta_1}{2}$ (see Fig. 35), and since θ_1 has been found, $\coth\theta_1$ may be deduced from a suitable table. We then have $\frac{R'_3}{\coth\frac{\theta_1}{2}}=Z_0'$.

The values of R and G may then be found as in previous cases, and the details of the uniform line to which Fig. 35 is equivalent determined. This uniform line, by comparison with the uniform lengths in Fig. 33, shows the effects of leak loading.

76. G. A. Campbell's Formula.—A definite formula showing the relation between a loaded section, such as is indicated in Fig. 28, and the uniform unloaded line to which it is equivalent, appears to have been first given by G. A. Campbell in an article entitled, "On Loaded Lines in Telephonic Transmission," *Philosophical Magazine*, vol. v. p. 319, March, 1903. The formula is as follows:—

$$\text{Cosh}\ \theta_1=\cosh\theta+\frac{Za}{Z_0}\sinh\theta$$

The symbols are here changed so as to be uniform with those given in this chapter. θ_1 is the line angle of a uniform unloaded line which is equal to the loaded section of uniform line whose angle is θ, and Za is the impedance of a load (Fig. 36).

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The formula may be proved as follows.

Let Fig. 36 represent one of the sections of Fig. 29, earthed at the receiving end, as shown.

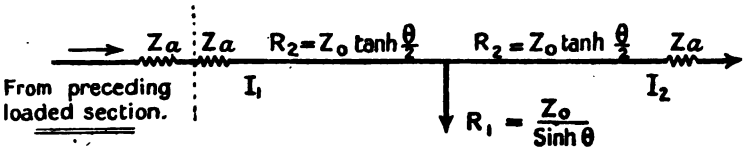

FIG. 36.

If I_1 is the current entering the loaded section, the current at I_2 is by inspection, by Ohm's law (see Chap. III.)—

$$I_2 = \frac{I_1 R_1}{R_1 + R_2 + Za}$$

By substitution of the values of R_1 and (R_1+R_2) from formulæ (2) and (4), Chap. III., we obtain

$$I_2 = \frac{I_1 \frac{Z_0}{\sinh \theta}}{Z_0 \coth \theta + Za} = \frac{I_1 Z_0}{Z_0 \cosh \theta + Za \sinh \theta}$$

Whence
$$\frac{I_1}{I_2} = \cosh \theta + \frac{Za}{Z_0} \sinh \theta \quad . \quad . \quad . \quad . \quad (8)$$

Now by (12), Chap. III., if I_1 and I_2 are the current at the beginning and end of a uniform line without apparatus, we have $\frac{I_1}{I_2} = \cosh \theta$. If the values of I_1 and I_2 are the same in the uniform line as in the case of Fig. 36 and formula (8), the value $\cosh \theta_1$ may be substituted for $\frac{I_1}{I_2}$, and we then obtain

$$\cosh \theta_1 = \cosh \theta + \frac{Za}{Z_0} \sinh \theta$$

Which proves Campbell's formula if I_1 and I_2 are the same in the case of the loaded and unloaded sections, which are also supposed to be of the same length.

77. Artificial Cables.—It has already been shown that any real circuit with uniformly distributed constants may be represented by an equivalent T or Π circuit for all electrical operations conducted from its two ends. There is no limit to the length of uniform line which may be so substituted in the direct current case; but in the alternating current case it will be seen later that limits are often imposed. It frequently happens that an artificial circuit or cable is required which represents a real circuit for a given length comprised between certain line angle limits, and is variable in steps of some definite line angle or angles between those limits. It is a common practice in such cases to utilize a number of nominal

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Π or T circuits in series, and by this means to make up any number of units of the nominal circuit, or some multiple of it, as may be required to form a complete artificial cable. We will now investigate the effect on the received current in general terms of utilizing a nominal T circuit instead of an equivalent T circuit (see Fig. 10, and the remarks concerning it in Chap. III.). The nominal T circuit in this case is given in Fig. 37.

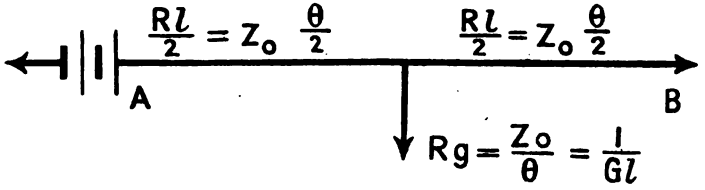

FIG. 37.

The apparent resistance of this circuit, measured from A, is

$$Z_1 = \frac{\left(\frac{Rl}{2}\right)^2 + 2\frac{Rl}{2}Rg}{\frac{Rl}{2} + Rg} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The received current at B is

$$I_2 = \frac{I_1 Rg}{\frac{Rl}{2} + Rg} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Now $I_1 = \frac{E_1}{Z_1}$. Substitute this value of I_1 in (10), and then substitute the value of Z_1 from (9), and we obtain

$$I_2 = \frac{E_1 Rg}{\left(\frac{Rl}{2}\right)^2 + RlRg} = \frac{E_1 \frac{Z_0}{\theta}}{\left(Z_0 \frac{\theta}{2}\right)^2 + Z_0\theta\frac{Z_0}{\theta}}$$

$$= \frac{E_1 \frac{Z_0}{\theta}}{\frac{Z_0^2\theta^2 + 4Z_0^2}{4}} = \frac{4E_1}{Z_0\theta(\theta^2 + 4)} \quad . \quad . \quad . \quad . \quad (11)$$

This expression involves an error due to lumping all the leakance at the centre of all the circuit resistance, instead of taking the correct equivalent T. The true received current at the end of a transmission line, or of an equivalent T earthed at the receiving end without apparatus, is given by formula (10), Chapter III., it is $I_2 = \frac{E_1}{Z_0 \sinh \theta}$. Fig. 38 shows the ratio between the received

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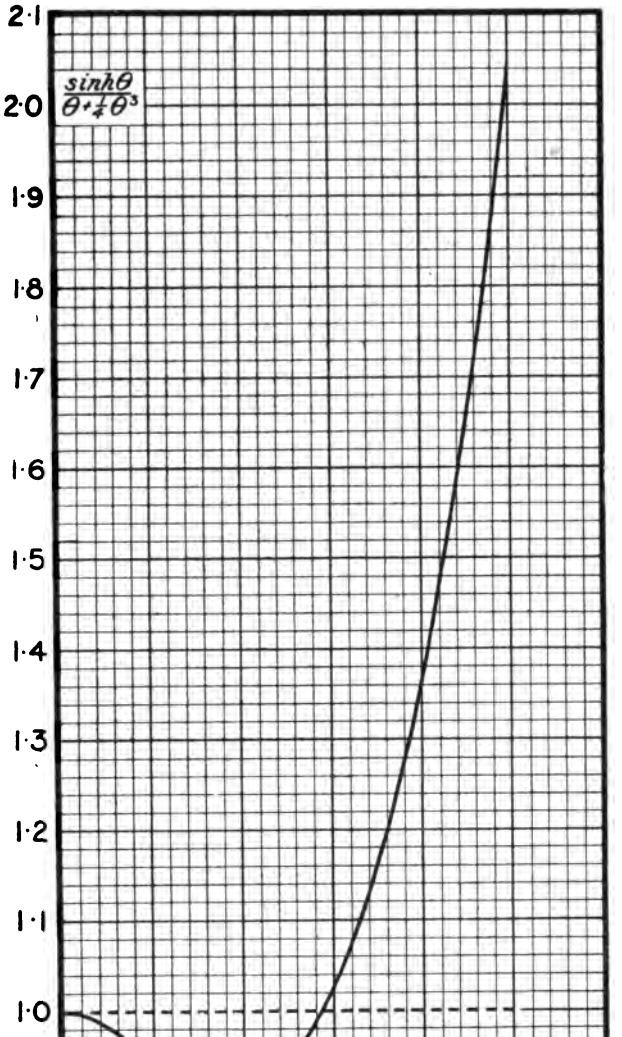

FIG. 38.

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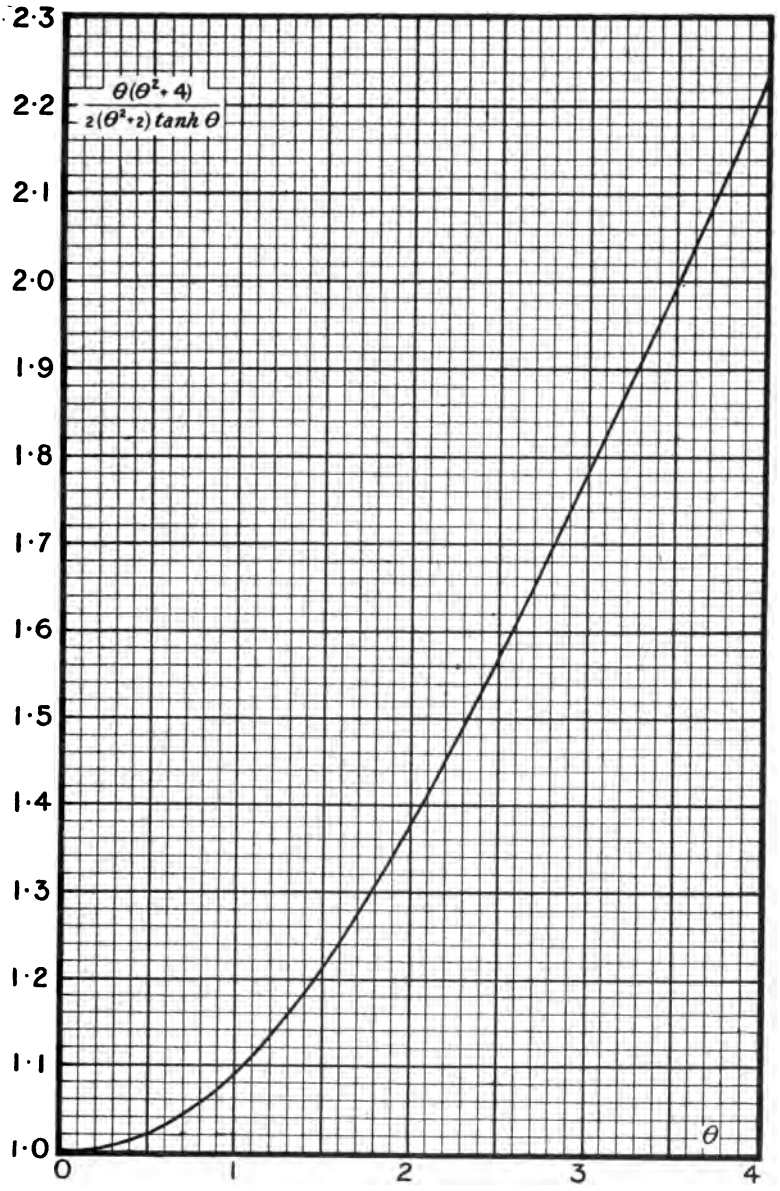

FIG. 39.

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current as given by formula (11) and the received current on the assumption that formula (10), Chap. III., the correct formula, is used. The ratio is

$$\frac{\frac{4E_1}{Z_0\theta(\theta^2+4)}}{\frac{E_1}{Z_0 \sinh \theta}} = \frac{4 \sinh \theta}{\theta(\theta^2+4)} = \frac{\sinh \theta}{\theta + \frac{1}{4}\theta^3} \quad . \quad . \quad . \quad (12)$$

The graph shows this ratio between the limits $\theta = 0$ and $\theta = 5$. It will be seen that for small values of θ there is practically no difference between the nominal T and the equivalent T. If, for example, the line angle is 0·2 the difference is less than half of one per cent., the error gradually increases with the line angle until it reaches one per cent. when the line angle has a value a little over 0·4. If, therefore, an artificial cable is divided into a number of small divisions each having a small line angle, the nominal T circuit will not materially differ from the equivalent T circuit, so far as the received current is concerned. It is of interest to note that when $\theta = 2{\cdot}86$ the current indicated by both the formulæ in question is the same. For greater values of θ the current obtained by the use of (11) becomes increasingly greater than that given by formula (10), Chapter III.

78. Error due to the use of the Nominal T Formula for the Measurement of the Sending-end Resistance of a Transmission Line with the Distant End Closed.—The relative formula is given by (13), which may be derived from Fig. 37.

$$Z_1 = \frac{\left(Z_0\frac{\theta}{2}\right)^2 + 2Z_0\frac{\theta}{2}\cdot\frac{Z_0}{\theta}}{Z_0\frac{\theta}{2} + \frac{Z_0}{\theta}} = Z_0\frac{(\theta^2+4)\theta}{2(\theta^2+2)} \quad . \quad . \quad (13)$$

This, again, contains an error due to lumpiness, the correct formula is $Z_0 \tanh \theta = Z_1$, from (6), Chap. III. The ratio of (13) to (6), Chap. III. has been plotted in Fig. 39 between the limits $\theta = 0$ and $\theta = 4$, and the error due to lumping all the leakance in the middle of all the resistance as shown in the nominal T may be traced by means of this graph.

79. Error due to the use of the Nominal T Formula in the Case of the Insulation Resistance, or, in other words, the Sending-End

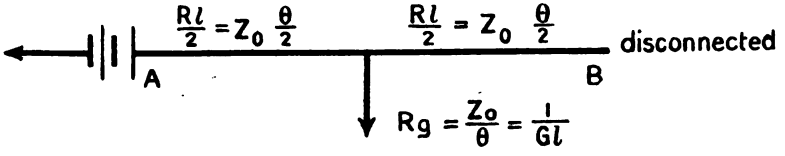

FIG. 40.

Resistance with the Distant End of the Circuit open.—The relative diagram is given by Fig. 40.

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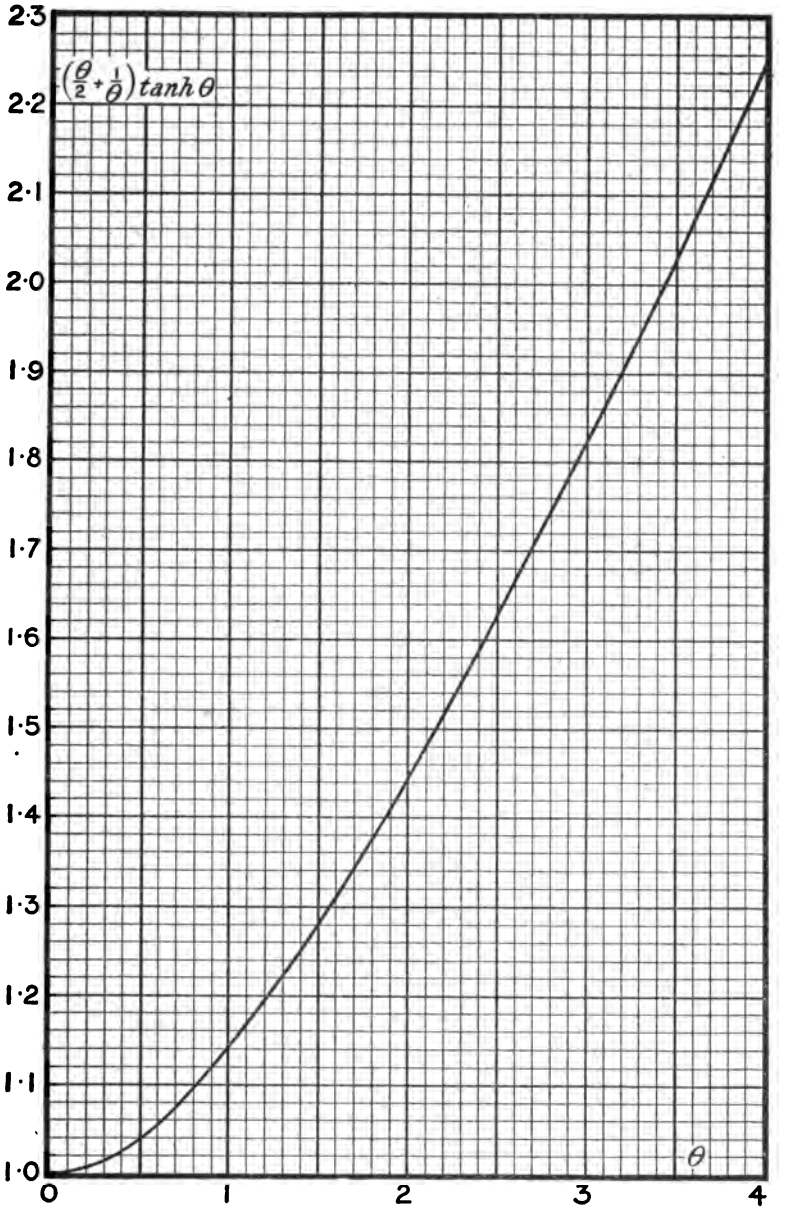

FIG. 41.

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It is evident from the inspection of this diagram that the resistance of the combination with B disconnected is

$$Z = \frac{Rl}{2} + Rg = Z_0 \frac{\theta}{2} + \frac{Z_0}{\theta} = Z_0 \frac{(\theta^2 + 2)}{2\theta} \quad . \quad . \quad (14)$$

The correct formula for this case is $Z = Z_0 \coth \theta$ by (5), Chap. III. The ratio between (14) and (5), Chap. III., is shown in Fig. 41. After having studied the preceding graphs the interpretation of Fig. 41 will present no difficulty.

It may be pointed out that the nominal diagram is not infrequently accepted as a correct means of investigating the insulation of an open telegraph line. Fig. 41 will show the extent of the error involved in this assumption (see p. 199). When once the correct data have been obtained and tabulated as here shown, the more exact method presents no more difficulty than the other, if a suitable table of hyperbolic functions is available. The formulæ developed in this chapter, however, find their more general application when alternating currents at telephonic frequencies are dealt with.

Example 1.—A uniform single wire circuit consisting of a 150-lbs. copper conductor having an insulation resistance of half a megohm per mile, is 10 miles in length, and has a series load of 30 ohms resistance at each end. If this loaded circuit forms one section of a uniform series loaded line, show what leak load must be placed at the two ends of a similar unloaded line to give the same line angle as that given by the series loaded line (see para. 73).

The uniform line with the loads at the end is shown by Fig. 42.

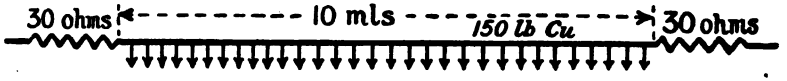

FIG. 42.

First calculate the equivalent circuit without the loads.

Now $$R = \frac{11{\cdot}73}{2} \text{ ohms per mile}$$

and $$G = 2 \times 10^{-6} \text{ mhos per mile}$$

$$\therefore \gamma = \sqrt{RG} = \sqrt{11{\cdot}73 \times 10^{-6}} = 0{\cdot}003425 \text{ hyp. per mile}$$

and $$Z_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{11{\cdot}73}{4 \times 10^{-6}}} = 1712{\cdot}4 \text{ ohms}$$

by (5) and (6), Chap. II.

Hence $$\theta = \gamma l = 0{\cdot}03425 \text{ hyp.}$$

The values of R_1 and R_2 of the equivalent T circuit of the line are

$$R_1 = \frac{Z_0}{\sinh \theta} = \frac{1712{\cdot}4}{0{\cdot}03426} = 49982 \text{ ohms}$$

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$R_2 = Z_0 \tanh \frac{\theta}{2} = 1712{\cdot}4 \times \tanh 0{\cdot}01712 = 29{\cdot}3$ ohms. (See Fig. 43.)

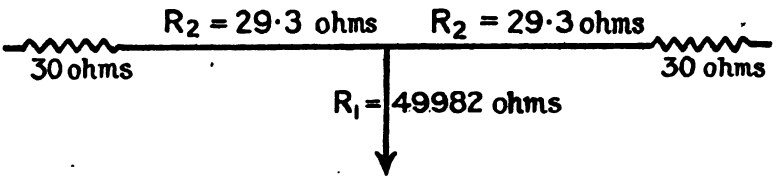

FIG. 43.

The values of R_1' and R_2' of the equivalent T circuit, including the series loads, are

$$R_1' = \frac{Z_0'}{\sinh \theta_1} = 49982 \text{ ohms}$$

(See the remarks relating to Fig. 29. See also Fig. 30.)

$$R_2' = Z_0' \tanh \frac{\theta_1}{2} = 29{\cdot}3 + 30 = 59{\cdot}3 \text{ ohms}$$

Now $$\cosh \theta_1 = 1 + \frac{R_2'}{R_1'} = 1 + \frac{59{\cdot}3}{49982} = 1{\cdot}001186$$

$$\therefore \theta_1 = 0{\cdot}0487. \quad \text{(See (36), Chap. III.)}$$

$$Z_0' = \frac{R_2'}{\tanh \frac{\theta_1}{2}} = \frac{59{\cdot}3}{\tanh 0{\cdot}02435} = 2435 \text{ ohms}$$

(See remarks after Fig. 30.)

The equivalent T circuit of the series loaded line is shown in Fig. 44.

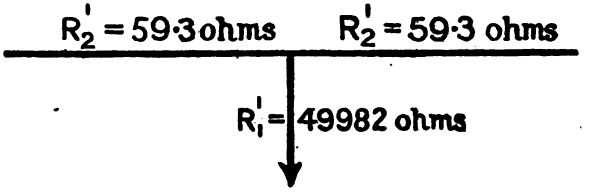

FIG. 44.

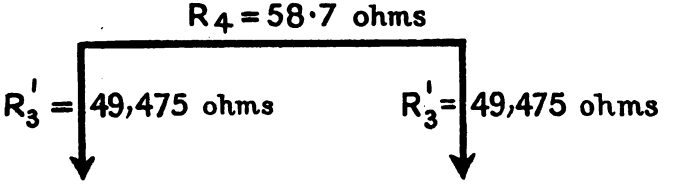

FIG. 45.

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The equivalent π circuit is found as follows:—

$$R_4 = Z_0 \sinh \theta = 1712{\cdot}4 \times {\cdot}03426 = 58{\cdot}67 \text{ ohms}$$

but $R_4 = Z_0'' \sinh \theta_1$ (See Fig. 32.)

$$\therefore Z_0'' = \frac{R_4}{\sinh \theta_1} = \frac{58{\cdot}67}{{\cdot}0487} = 1205 \text{ ohms}$$

$$R_8' = \frac{Z_0''}{\tanh \frac{\theta_1}{2}} = \frac{1205}{{\cdot}02435} = 49{,}475 \text{ ohms} \quad \text{(See Fig. 45.)}$$

$$\left[\text{See (4). Note that } Z_0'' \coth \frac{\theta_1}{2} = \frac{Z_0''}{\tanh \frac{\theta_1}{2}}\right]$$

$R_8' = 49475$ ohms is the leak load required.

Example 2.—(*a*) If the series load of 30 ohms had been distributed uniformly over the line circuit, what would be the resistance of the latter?

(*b*) Find the resistance of a uniform circuit without load which has the same line angle as the loaded uniform circuit.

(*c*) Find the received currents for the unloaded circuit and for the equivalent series and leak loaded circuits when an E.M.F. E_1 is applied at the sending end.

(*a*) $Rl = \frac{11{\cdot}73}{2} \times 10 + 60 = 58{\cdot}65 + 60 = 118{\cdot}65$ ohms.

(*b*) $Rl = Z_0'\theta_1 = 2435 \times {\cdot}0487 = 118{\cdot}64$ ohms.

(See notes relating to Fig. 29. Z_0' and θ_1 are known from the preceding example.)

(*c*) Received current with unloaded circuit; Z_0 and sinh θ are known from the preceding example.

$$I_R = \frac{E_1}{Z_0 \sinh \theta} = \frac{E_1}{1712{\cdot}4 \times \sinh {\cdot}03425} \text{ (by (10), Chapter III.)}$$

$$= \frac{E_1}{1712{\cdot}4 \times {\cdot}03426} = \frac{E_1}{58{\cdot}67} \text{ amps.}$$

Received current with series loaded circuit

$$I'_R = \frac{E_1}{Z_0' \sinh \theta_1} = \frac{E_1}{2435 \times {\cdot}04872} = \frac{E_1}{118{\cdot}6} \text{ amps. [See formula (2).]}$$

Received current with leak loaded circuit (Z_0'' and sinh θ_1 are known from the preceding example)

$$I_R'' = \frac{E_1}{Z_0'' \sinh \theta_1} = \frac{E_1}{1205 \times {\cdot}04872} = \frac{E_1}{58{\cdot}67} \text{ amps.}$$

It will be seen from the last example under (*a*) and (*b*) that if a suitable series load is placed at each end of a transmission line

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with a small line angle practically the same effect is produced as if the resistance had been uniformly distributed over the whole length. This, however, is only true for relatively small line angles. In view of the fact that the resistance of half an arm of the equivalent T is equal to $Z_0 \tanh \frac{\theta}{2}$, it will be evident that for large values of θ a large change of resistance produces very little effect on the total resistance of the arm of the equivalent circuit, seeing that $\tanh \frac{\theta}{2}$ attains a maximum value of 1 when θ is large. In the case of a real transmission line the resistance is always $Z_0\theta$ with any length of line. It follows that lumped loads only approximate in effect to uniformly distributed ones when the line angle is small, this is important in considering the effect of the spacing of coils in the loading of telephone lines.

The results obtained under (*c*) bring out the fact that the attenuation constant is increased by series loading in the direct current case and that the received current of the loaded circuit may be reduced as compared with the unloaded one. This depends on the relative values of Z_0 and the circuit length. The factors which cause an increase in the attenuation constant are also present in alternating current loading; but in that case they are accompanied by the beneficial effect of inductance on capacity, which usually produces a decided net transmission gain if the circuit is sufficiently long.

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CHAPTER V

THE HUMAN VOICE IN TELEPHONY

80. Introductory Remarks.—It is generally assumed in the study of telephonic transmission that the voltages and currents generated in the telephone transmitter by the voice, and transmitted along telephone circuits, are alternating currents and voltages of sine wave formation. The vibrations which are evident in the production of a given speech sound are supposed in fact to be capable of resolution into a number of different wave forms, each of which is supposed to recur often enough to impress its character on the circuit. When this regularly recurring state is reached the resulting voltages and currents are said to have attained the steady state. This state is not arrived at instantaneously, but there is some small period during which the voltage is being built up, before it arrives at the steady state, or, on the other hand, there may be a small period when the voltage and current are changing from the steady state to zero. Electrical phenomena which are manifested in this period are known as transient phenomena, and in telephony this period is supposed to be relatively small and unimportant in comparison with the duration of the steady state. The sounds of the human voice are, moreover, made up of a large number of frequencies, and the vibrations constituting a given speech sound are *simultaneously* impressed upon the telephone transmitter and traverse the line. It is generally supposed that the mean frequency of speech may be taken for telephonic purposes as being somewhere in the neighbourhood of 800 periods per second. Before proceeding to the detailed study of the application of the assumptions on which the laws of telephonic transmission are based we will now examine some of the evidence relating to the matter.

In order to obtain a clear idea of the problem it will be advantageous to first briefly examine the mechanism of speech, otherwise the significance of the speech records cannot be fully appreciated.

81. The Organs of Speech.—The principal organs of speech are the vocal cords in association with the mouth and the nose.

The Vocal Cords are fixed in the windpipe where the "Adam's apple" is seen. In the position of rest the breath passes unobstructedly through the glottis or opening between the vocal cords. When these latter are lightly drawn together and air passes out, the cords vibrate, and a "voiced" sound is the result; such sounds are an essential part of all vowels, and some consonants, and they can be varied in pitch. After the breath has passed through the vocal cords it may pass through the mouth or the nose, or both, and the sound may thereby be modified. Sounds modified

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in the nose are nasal consonants, such as *m*, *n*, *ng*. In passing through the mouth the voice may be modified in various ways, due to the movements of the jaws, lips, and tongue. If the passage of the breath through the mouth is free, after a vibration of the vocal cords, the sound produced is a vowel, if not free it is a consonant; for example, if the vowel *ah* be pronounced, a continuous sound is produced in the vocal cords, and the shape of the mouth is modified for the production of the sound, but at the same time the mouth is open and there is a free passage of air through it. In the pronunciation of a consonant, however, for example *p*, the free passage of the air is temporarily stopped, in this case by the lips. The oscillograph records which follow later on have been selected as characteristic examples of the sounds produced in the vocal cords, the mouth, and the nose, and as sufficiently representing the records of the mechanism of speech for the purpose in view.

82. Nature and Varieties of Sound.—All sound is caused by the vibratory motion of bodies; when these vibrations reach the ear they are recognized as sound. The vibratory nature of sound is easily made evident; for instance, if the vowel *o* be pronounced loudly, and the lips lightly touched during the process, the vibration may be felt. The maximum distance through which vibrating particles of a sounding body moves is called its amplitude, and the greater the amplitude the louder the sound, or the greater its *intensity*. Two sounds may also affect the ear differently, in proportion to their rapidity of vibration or greater frequency: the more rapid the vibration the shriller the note, or the higher is its *pitch*. Two notes of the same intensity and pitch may produce different impressions on the ear, for instance, the same note on a piano and a violin; this difference is due to the different *timbre* of the two sources of sound. It is caused by the fact that each of the two fundamental tones is accompanied by others of different pitch superimposed on it; these are overtones, and they are usually of different amplitude to the fundamental, and constitute a different series in each case.

Sound is broadly divided into two categories, viz. noise and musical sounds; the former are either abrupt sounds or non-periodic complex vibrations, or periodic discords, whilst the latter are the result of the continued periodic vibration of the particles of sounding bodies, whose periods are appropriately related. Periodic vibrations are assumed in sine wave transmission.

83. Resonance.—A good tuning fork has a natural period at which it vibrates and gives its characteristic tone. If another body in the neighbourhood makes the same number of complete vibrations per second, either as fundamental or overtone, it will communicate its vibration to the tuning fork if it be at rest, and if the amplitude of the vibrating body is sufficient. The reason of this phenomenon is that each time the tuning fork makes a complete vibration, and is about to repeat the motion, an impulse arrives from the vibrating body which assists the motion of the tuning fork, and this is repeated at the end of every complete period: at the commencement the motion of the fork may be extremely small; but the accumulation of impulses in the same direction increases the amplitude up to a certain point and maintains

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it afterwards. Any body having a natural period may be similarly affected. An analogous phenomenon may be observed in connection with the vibration of air columns in tubes; if, for example, a vibrating tuning fork is placed near the opening of a tube which is closed at the other end, and if the tube be of such a length that its fundamental coincides with the tone of the fork, the note will be strengthened; if the wave length in air of the tone of the fork is λ the length of a closed pipe which gives this as its fundamental is $\frac{\lambda}{4}$.

Resonance plays a part in the formation of articulate sounds; this modifying resonance is effected, for example, by the cavity of the mouth and the nasal cavity. Resonance also comes prominently into play in telephonic apparatus.

84. Limits of Pitch of the Human Voice.—The voice can only cover a portion of the possible range of musical sound. Male singing voices are usually comprised between the limits of 80 and 400 to 500 periods per second, and female voices between 160 and from 700 to 800 periods per second, although there are exceptions to these rules, and different authorities do not always agree on the precise figures; the tones referred to are fundamentals, and they are generally accompanied by a number of overtones.

85. Mean Frequency of the Voice in Telephony.—In dealing with the apparent frequency of the human voice as deduced from telephonic experiments, the frequency of the fundamental is not the one in question; the sounds comprised in the human voice cover a number of different fundamentals (for example, males and females are different), and these fundamentals are in all cases associated with overtones; these facts are corroborated by the oscillograms which are here shown. In practice it is found that the combined frequencies which are made up of the fundamentals and overtones which are simultaneously transmitted along telephone circuits, are subjected to the same attenuation as some one tone of a definite frequency intermediate between the lowest and the highest. The question of the mean frequency of speech was dealt with by the second Conference Internationale des Techniciens des Administrations des Télégraphes et Téléphones de l'Europe, held in Paris in 1910, where it was unanimously decided (*Comptes Rendus*, pp. 99–100) as follows: "In practical calculations, the telephonic current may be replaced by a current of sine wave form. As regards intensity the angular velocity of $2\pi f = 5000$ may be adopted as a mean figure. It is desirable in addition to take the angular velocities of $2\pi f = 3000$ and 7000 into account so far as timbre is concerned." The mean frequency adopted is practically 800 periods per second, for $2\pi \times 796 = 5000$, and the latter figure is much more convenient for calculation than if the exact frequency of 800 were taken.

86. Range of Sound perceived by the Ear.—The ear perceives sounds over a much wider range than those produced by the voice, the upper and lower limits of pitch discerned by the ear are in the neighbourhood of 16 and 40,000 according to some authorities; the range varies, however, with different persons. The sensitivity of the ear is greater for relatively high speech frequencies than for

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very low ones. According to Lord Rayleigh (*Phil. Mag.*, p. 603, 1907), the maximum sensitiveness of the ear is reached at not less than 1024 vibrations per second, and possibly higher.

87. The Oscillograph.—In order to give an adequate idea of the nature of telephonic speech, and to obtain information as to the nature of the problems outlined at the beginning of this chapter, a number of oscillograph records taken in the Laboratory of the Engineer-in-Chief, G.P.O. West, London, are here given by permission. The oscillograph consists of two separate insulated conducting loops, capable of movement between the poles of a powerful electromagnet. When either or both of the loops are connected to separate electric circuits, and a current circulates in the loop, it is deflected, the amplitude of the deflection being proportional to the strength of the current in the loop. The suspended loops are so designed that their movements accurately reproduce the wave form of the currents passing through them, these movements being indicated by means of the reflected beams of light from a

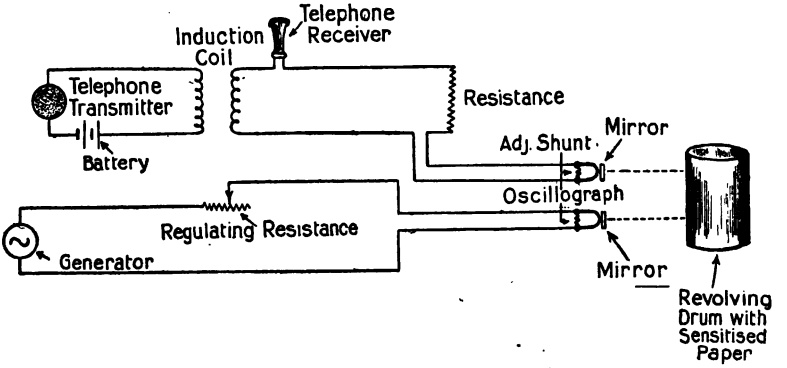

FIG. 46.—Arrangement for recording oscillograms.

small mirror attached to them. The reflected rays fall on moving photographic paper fixed on the surface of a rotating drum placed in the path of the rays (Fig. 46). The photographic paper is of course shielded from all light except that reflected from the mirrors. The functions of the two circuits connected to the oscillograph are as follows. In the telephonic circuit the receiver at the receiving end of the line is replaced by the oscillograph suspension, and any sounds spoken into the transmitter cause telephonic currents to pass through the oscillograph loop and cause it to be deflected in proportion to the instantaneous strength of the current passing through it, the resulting wave form is recorded as already explained. In order to permit of the number of vibrations per second of the speech curves being readily and accurately known a generator circuit was employed. The generator used was an alternating current machine revolving at a fixed speed (giving generally 800 periods per second), and it was possible at any instant to measure this speed accurately. The generator currents passing through the oscillograph loop were recorded simultaneously on the same paper

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as the telephonic voice currents. By this means, knowing the speed of the generator, it was possible to determine the frequency of the voice currents. The oscillograph used was a "Blondel" oscillograph in most of the cases, but some of the records were taken with a "Duddell" oscillograph. In some tests the two oscillograph loops of the Blondel oscillograph were placed at the beginning and end of a telephone line instead of being arranged as shown in Fig. 46, which arrangement permits of the wave form and amplitude at the sending and receiving ends of a given circuit being investigated. A time scale and a current scale are shown on the oscillograms where these details are known; but in some instances they were not recorded.

88. Examination of the Oscillograms. General.—Before examining the telephonic speech records, some limitations of the method, and of the telephone as an accurate means of reproducing human speech, will be pointed out.

(*a*) In the first place, the oscillograph as at present constructed is a much less sensitive instrument than the telephone receiver. It is possible to hear the acoustic effect of less than one millionth of an ampère in a good commercial telephone receiver, and this is actually required for some delicate cross-talk tests. In the case of the oscillograms, a reference to the current scale marked on a number of them, will show that 0·45 inch or 1·195 cm. represents the amplitude of the oscillograph deflection for 10 milliampères. Now commercial speech can easily be conducted if 1 per cent. of this current is available at the receiving end of a telephone circuit, but on the contrary an oscillograph deflection of 1 per cent. of 10 milliampères would only have an amplitude of 0·0045 inch or approximately 0·01 cm., which is merely the thickness of a line, so that details could not be distinguished in such a case. It is possible, therefore, that some of the speech overtones of very small amplitude may not be recorded in the oscillograms.

(*b*) Certain consonants, such for example as *s* and *sh* (as in *ish*), are not perfectly transmitted by ordinary telephones, and the records of such letters may be expected to show a corresponding degree of imperfection.

(*c*) Speech records are more or less modified by variations in the construction of different types of telephones, and the nature of the load connected to the telephone, *i.e.* according to whether the load is made up of ohmic resistance, capacity, or inductance, or some combination of these. As regards the transmitting portion of the telephone, it is known that the higher frequencies are subjected to a greater impedance than lower ones; for example, the transmitting circuit used for the greater part of the records here given showed a variation of approximately 10 per cent. in impedance between the limits $2\pi f = 3000$ and $2\pi f = 7000$ with the microphone replaced by a constant resistance equal to its mean speaking resistance. When, however, the microphone was placed in the measuring circuit its resistance introduced a variable element which masked to a large extent the effects of the frequency on the impedance. It is known also from recorded experiments,[1] that com-

[1] Bela Gati, *Electrician*, Dec. 30, 1910. Meyer and Whitehead, *Transactions* American Institution of Electrical Engineers, June, 1912.

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mercial types of telephone diaphragms are subject to resonance; one prominent resonance point being in the neighbourhood of the frequency selected as the mean frequency of speech, viz. 800 periods per second. Notwithstanding these factors a comparison of some of the records with others taken by entirely different methods shows a general resemblance to those here given.

The speech records here shown illustrate the complicated wave forms which are transmitted along telephone circuits, subject to the limitations of the sensitivity of the recording apparatus, as already pointed out. All the records are those of male speakers. Figs. 47, 49, 50, 51, and 54 to 58 inclusive are voice records of one speaker, and the rest, up to 65 inclusive, of the other. A few records of a female speaker were taken, but the records were failures. In the case of the records on which a time and current scale is marked, care was taken to ensure that the speech recorded represented normal pitch and intensity, except in Figs. 56 and 58, representing speech over a line with an attenuation constant of $\beta l = 2{\cdot}1$. In this case it was necessary to speak very loudly.

89. Vowel Records.—Vowels enter into the composition of all the letters of the alphabet, and three representative vowels are given, viz.: (Figs. 47 to 51) *a* as in pay, *ee* as in see, and *o* as in go. In these three cases the record shows a sustained vowel sound of a definite wave form, such that if the nature of a given vowel curve is once noted, it can usually be identified in combination with other sounds in more complicated records taken in the same circuit. The vowel *e* enters into the composition of the letters *p* and *b*, for example, and these letters can be compared with the vowel record. The vowel record in each case consists of a series of similar and symmetrical wave forms which vary somewhat with different individuals. It was noticed, however, in taking the curves, that if the amplitude of the current waves for the same letter are widely different in any two cases, the resemblance is much less marked. (See, for example, Figs. 56, 57, and 58, which are records of loud speech.) In order that the curves may be strictly comparable it is desirable that the same poles of the battery should be connected uniformly to the same oscillograph terminals, so as to circulate the current in the same direction around the oscillograph loop; if this is not done the record appears to be reversed in one case as compared with the other; for example, the vowel portion (*e*) of the letter *b* in Fig. 54 appears to be reversed as compared with the vowel *e* in Fig. 47 for the reason stated. Only one record is available in the case of the letter *a*, but in the case of *e* two records, and for *o* three oscillograms, obtained from different speakers, are given. The regularly recurring vowel wave forms in each case are produced by a fundamental tone and its harmonics, each of these wave forms according to the theorem of Fourier may, in fact, be replaced by a fundamental sine wave, and a greater or less number of harmonics, and these sine waves taken together have the same resultant effect as the wave form in question.

90. The Time required for Telephone Current Waves to attain the Steady State (see Para. 80).—In order to properly appreciate the evidence furnished by the oscillograms on this point, it is necessary to examine one or two records in detail, and Figs. 52

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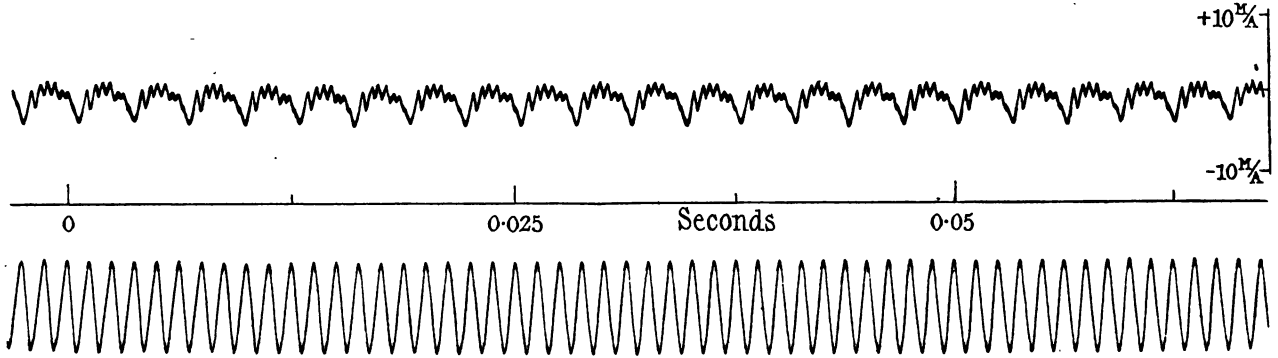

FIG. 47.—E, as in "see."

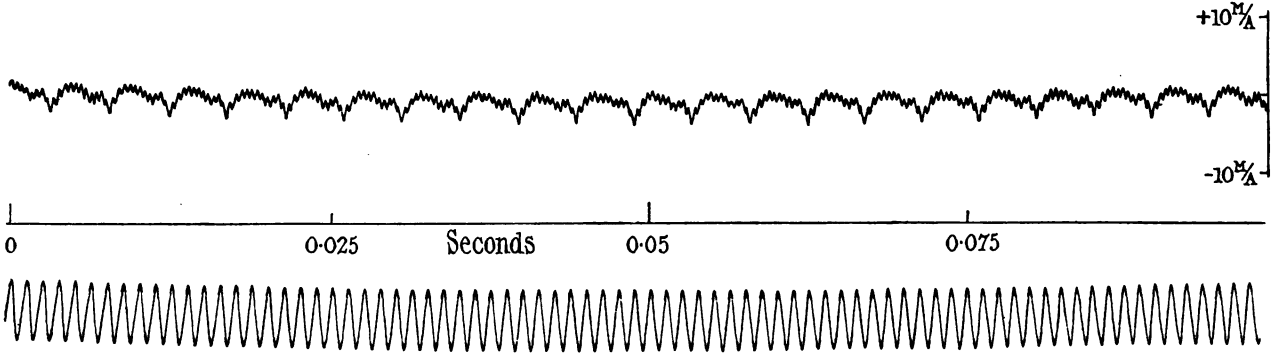

FIG. 48.—E, as in "see."

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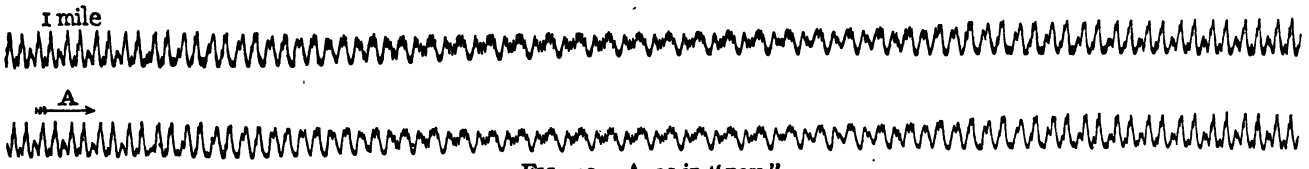

FIG. 49.—A, as in "pay."

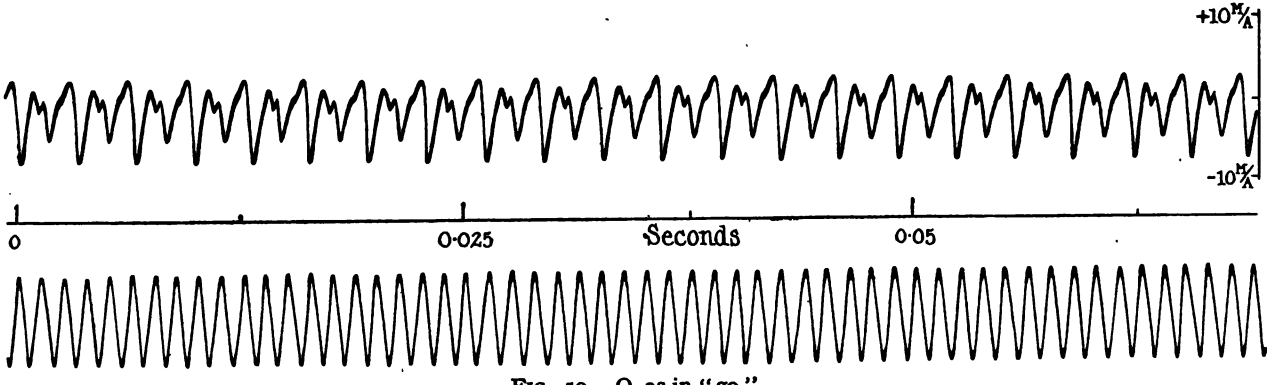

FIG. 50.—O, as in "go."

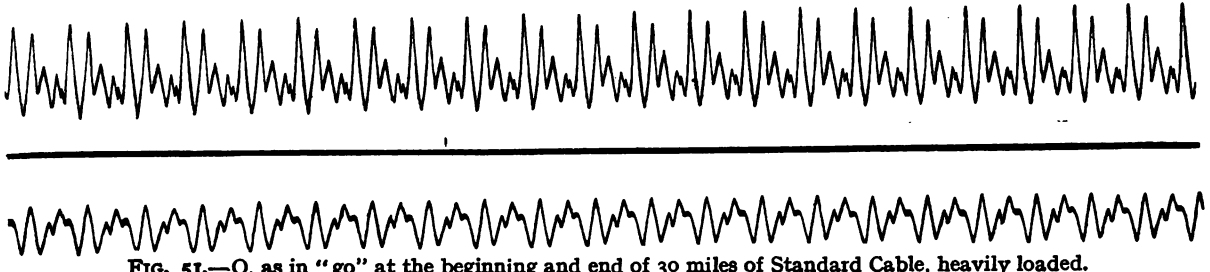

FIG. 51.—O, as in "go" at the beginning and end of 30 miles of Standard Cable, heavily loaded.

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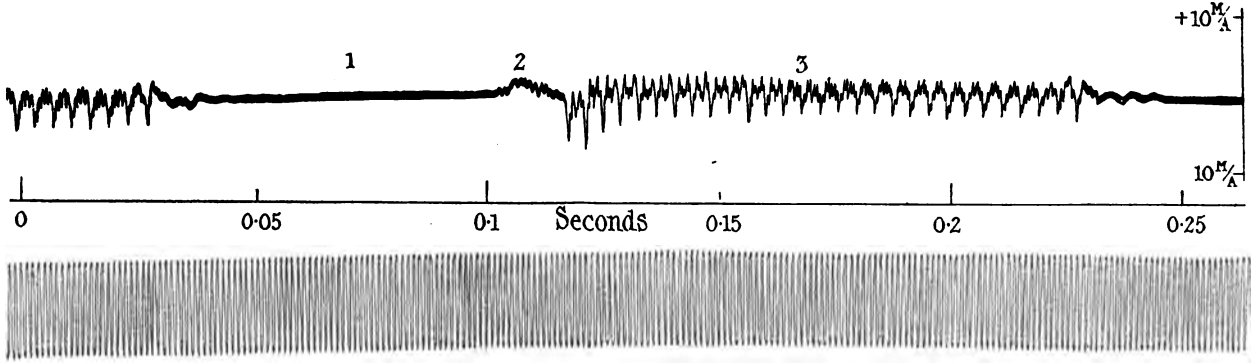

FIG. 52.—P, pronounced "pea."

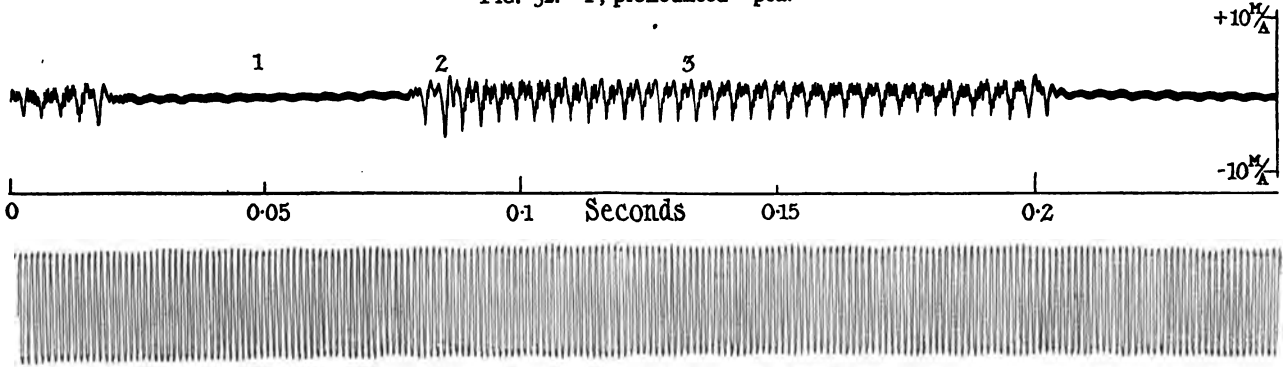

FIG. 53.—B, pronounced "bee."

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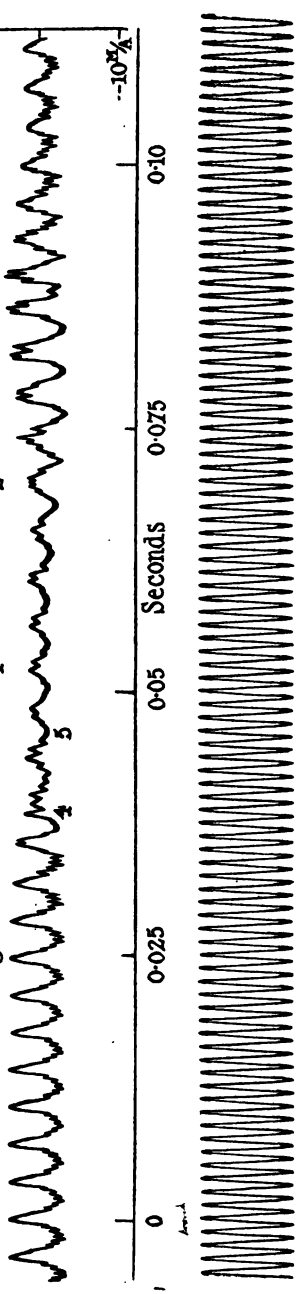

FIG. 54.—B, pronounced "bee."

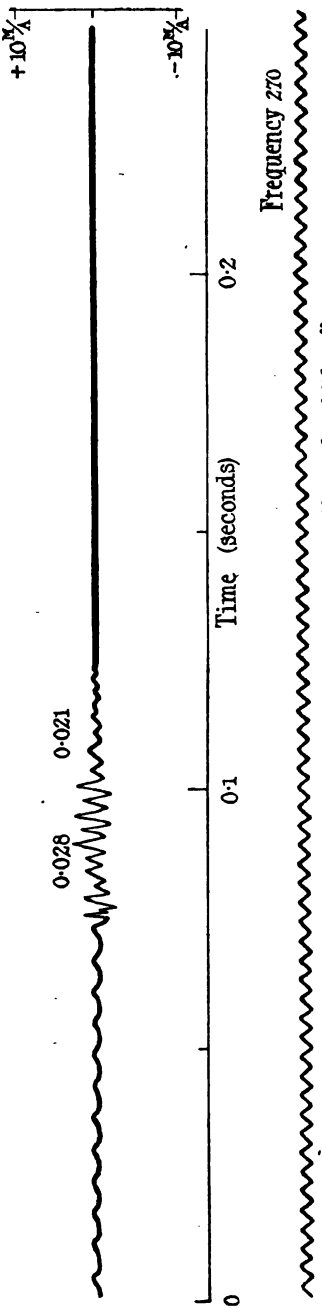

FIG. 55.—This figure shows the consonant portion only of "bee."

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and 53 are good examples. A current scale is marked at the right-hand side of the figures and a time scale is given below the speech oscillograms. The time scale is constructed from the alternator record at the bottom of the figure, which is the curve of a sine wave current having a frequency of 800 cycles per second. The letter *p* is made up of an explosive consonant followed by the vowel *e* (as in pea). In Fig. 52 the parts are numbered 1, 2, and 3. Here 1 indicates an interval of silence between two records of the same letter, 2 is the consonant, and 3 the vowel. So far as can be judged from the record the whole of the explosive part of the consonant only occupied about 0·015 second in pronunciation, and it appears to change gradually into the vowel *e*, as may be seen by identifying the outline of *e* by comparison with Fig. 47, and yet a speech trial with the telephone used in making the oscillogram showed that the letter could be generally recognized at the end of 46 miles of standard cable, although it was sometimes confused with *b*. The current received at the end of 46 miles of standard cable is only 0·75 per cent. of the current at the sending end of the same cable. Fig. 53 shows the letter *b*: this is distinguished from *p* by the fact that the explosive consonant at 2 is preceded by a voiced sound, produced by the vibration of the vocal cords, as may be readily verified by pronouncing the letter. The parts 1, 2, and 3 show respectively the voiced sound, the explosive sound caused by the lips, and the vowel. In this case, however, the explosive is hardly distinguishable on the oscillogram. The details of this letter are better seen in Fig. 54, which is the record of a different speaker; in this case, however, the oscillograph drum was run much more rapidly. In this record the fig. 3 at the left-hand side of the oscillogram indicates the vowel *e* at the end of one letter *b*, and the fig. 1 shows the voiced part at the commencement of another letter *b*. The speaker whose voice is recorded in this case pronounced the letter a number of times in rapid succession, so that the end of one letter is practically continuous with the beginning of the next, the total time of transition from the last full period of the *e* in 3 at the left-hand side to the first properly formed period of the voiced part of the following letter, as indicated by the figs. 4 and 5, is not more than one-hundredth of a second, and it may be less, because it is possible that the transition curve between 4 and 5 was largely due to modification of the voice in changing from the end of one letter to the beginning of the next, rather than to the time required to change the current to the steady state. The line in this case was, however, made up of resistance only. Fig. 55 gives the consonant part of the *b* only (*i.e.* it gives the *b* without the final *e*) by the same speaker as Fig. 54, and Fig. 47 shows the vowel *e* taken in the same conditions. Fig. 56 shows a record of the letter *b* at the beginning and end of a line of 20 miles of standard cable (equal to $e^{-2\cdot1}$), the speaker was the same as in the case of Fig. 54, the apparatus, including the oscillograph, was also the same in the two cases, and there is a resemblance in the general character of the curves, if the different nature of the line is taken into account. A speech test in the case of this letter *b* showed that it could be distinguished in the same conditions as the letter *p* already referred to, over 46 miles of the standard cable. A reference to the Figures

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will show that the *maximum* amplitude of the current where recorded did not exceed 10 milliampères at the sending end of the line in the case of *p* and *b*. It is known that a current whose root mean square is 5 milliampères represents fairly loud speech, and that normally such a current is not often exceeded in telephonic conversation.

The foregoing facts, then, show that the explosive consonant part of a letter *p* or *b* represented by a current of not more than 10 milliampères at the sending end, and sometimes lasting not longer than two-hundredths of a second as a maximum, may be recognized by means of a telephone receiver at the receiving end of a transmission line having an attenuation constant such that the received current is less than one per cent. of the sent current. The failure to distinguish a letter *p* from a *b* is probably not due to any failure in recognizing the explosive consonant, but is more likely to be due to the fact that the voiced consonant preceding the explosive one in the letter *b* is not always audible, and it will be observed from the records that the amplitude of the voiced consonant is less than that of the explosive one. It appears probable, therefore, that amplitude is more important than length of signal, and that the duration of an explosive consonant, which represents one of the shortest speech sounds, is sufficient to permit of its transmission over a long line with sufficient amplitude to ensure its recognition in a telephone receiver at the end of the line.

91. Comparative Amplitude of Speech Sounds as shown by Oscillograms.—It will be at once evident by inspection of the oscillograms (Figs. 56 to 65), that the amplitude of the component parts of the same oscillogram are in many cases very different; this remark applies to any of the records of a complete letter made up of a vowel and consonant, for instance *p*, *b*, *f*, *s*, *ish* and *zhee*. This is very noticeable in the record of the letter *f* (Fig. 59). The oscillogram in this instance only shows the vowel portion of the letter at its commencement, although the letter was clearly pronounced in a normal manner; when, however, a speaking test was made by the same apparatus and speaker as recorded in the oscillogram, the consonant sound missing in the oscillogram was audible in a telephone receiver at the end of 20 miles of standard cable. In order to obtain a record of this sound it was afterwards uttered as loudly as possible in a continuous manner without the vowel part, the result is shown in Fig. 60. The record of letter *s* (Fig. 61) also fails to show the hissing consonant portion of the letter, the recorded portion shows only the vowel at the beginning of the letter, and this is similar to that at the beginning of the letter *f*. When a speech test of the letter *s* was made in the same conditions as before, it was found that the hissing sound was not at all faithfully recorded in the receiver, and when the latter was joined to the transmitting apparatus direct without cable, the same imperfection was noticed; it was in fact found that the received sound resembled *f* more nearly than *s*, and it was difficult to distinguish these letters from each other; other telephone sets gave the same result; the telephone apparatus does not in fact appear to be capable of correctly reproducing this letter. The difficulty experienced in its transmission is, of course, well known. A separate record of the

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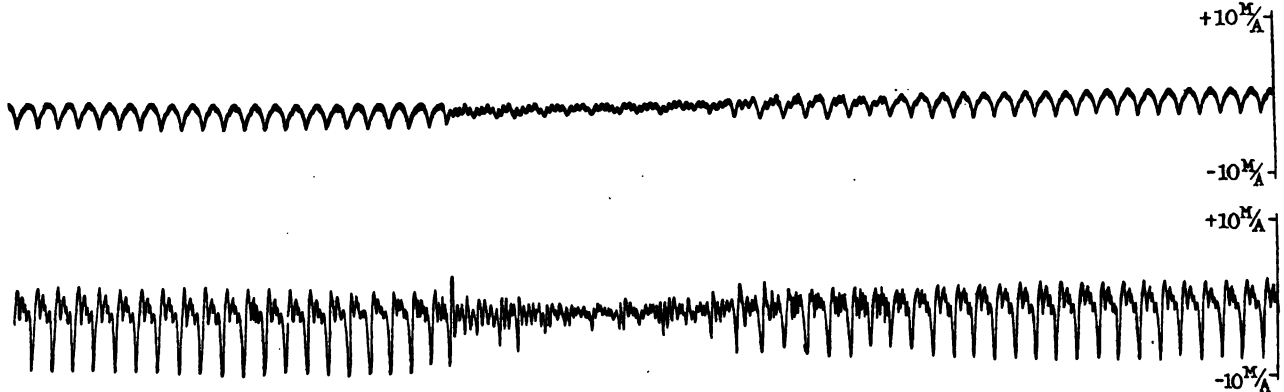

FIG. 56.—B, pronounced "bee," at the end and beginning of 20 miles of unloaded Standard Cable.

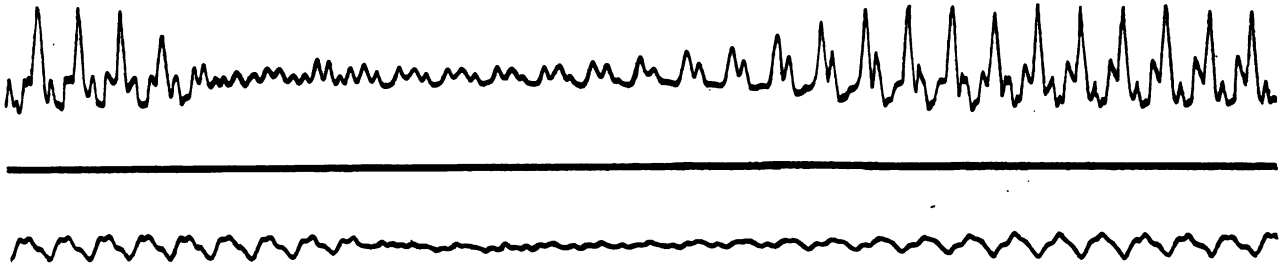

FIG. 57.—B, pronounced "bee," at the beginning and end of 96 nauts of loaded Submarine Cable.

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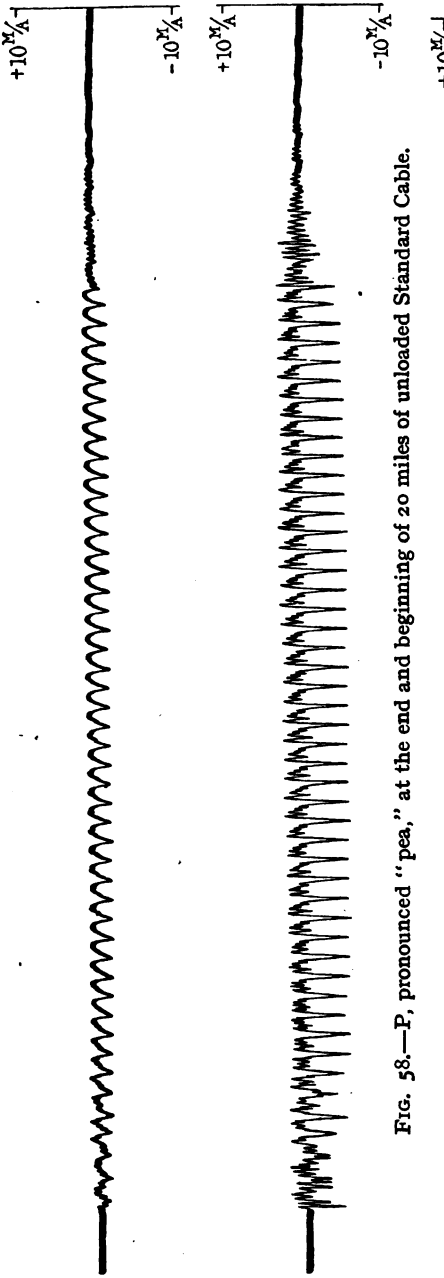

FIG. 58.—P, pronounced "pea," at the end and beginning of 20 miles of unloaded Standard Cable.

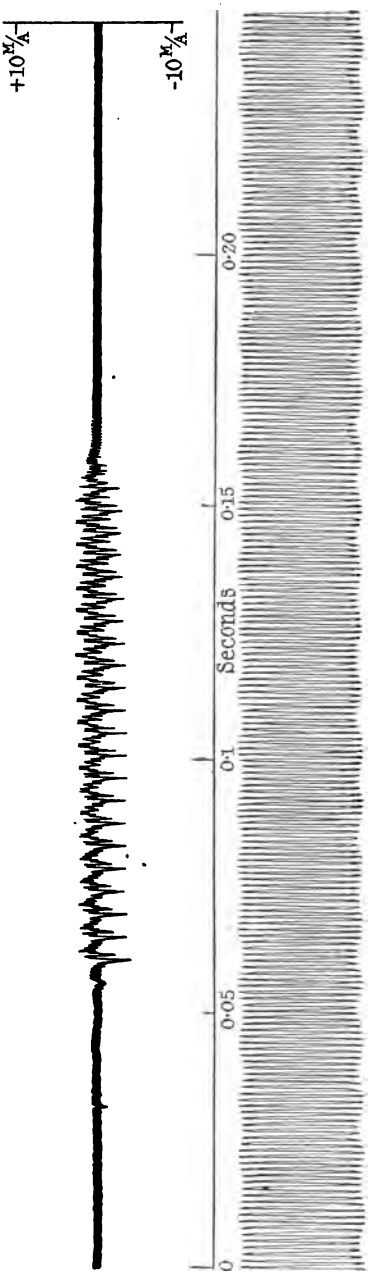

FIG. 59.—F, pronounced "eff."

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FIG. 60.—This shows the consonant portion only of F.

FIG. 61.—S, pronounced "ess."

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FIG. 62.—This shows the hissing portion only of S.

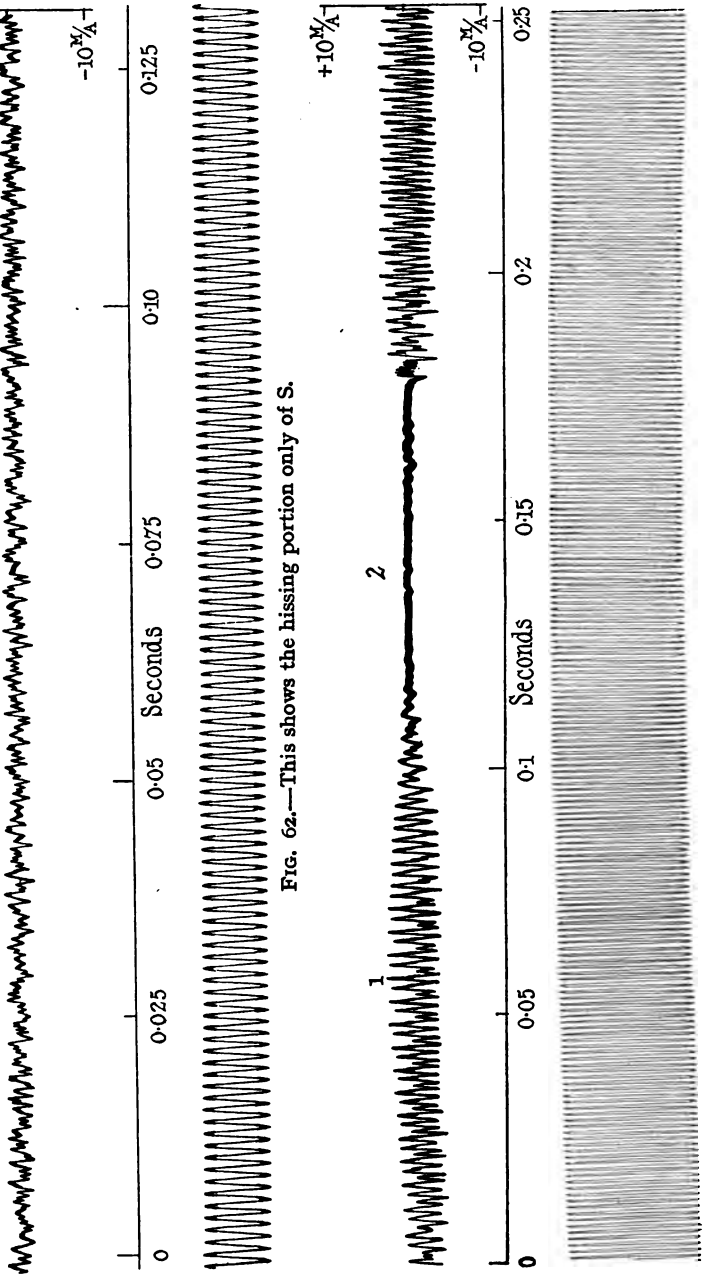

FIG. 63.—"Ish."

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hissing portion of the *s* is given in Fig. 62. A marked disproportion in the amplitude of the vowel and consonant portion of the sound *ish* is seen in Fig. 63, and a separate record of the consonant portion *sh* is given in Fig. 64. An attempt to record the sound *zhee* is shown in Fig. 65. In all the cases examined the vowel gives generally a much greater amplitude than the consonant except in the case of the explosive consonants *p* and *b*, which are very short. Now when the consonant sounds are associated with a relatively high frequency as in the cases of *f*, *s*, and *ish*, the imperfect rendering of such sounds in long distance transmission is likely to result. If ordinary conversation is conducted in such cases, however, by practised speakers, familiarity with the language usually enables the mind to fill in the gaps; but if figures or separate letters are transmitted it is sometimes necessary to resort to well-known mnemonic devices to secure accurate transmission.

92. Variation of the Wave Form with the Type of Circuit over which the Speech Waves are transmitted.—The oscillograms of telephonic sounds vary considerably with the nature of the load connected to the telephone. The three examples of the letter *b* (Figs. 54, 56, and 57) illustrate this. Fig. 54 is the record of the letter *b* when the circuit in series with the telephone and the oscillograph consisted only of 200 ohms ohmic resistance. In Fig. 56 the ohmic resistance was replaced by an artificial cable equivalent to 20 miles of standard cable (Chap. XI.). In Fig. 57 the circuit connected to the telephone and oscillograph was a loaded submarine cable having 100 millihenrys inserted at intervals of one naut; the circuit length was 96 nauts. This record differs in wave form from the other two. Figs. 50 and 51 are oscillograms of the letter *o*, the speaker being the same in each case. In the case of Fig. 50 the circuit consisted of 200 ohms, whereas in the case of Fig. 51 speech was conducted through 30 miles of loaded standard cable. This record shows the sent and received currents at the ends of the cable when the vowel *o* was spoken; the cable was loaded with 250 millihenrys at intervals of one mile. It will be seen that the effect of loading on the amplitude is marked. This is seen by comparing this record with Fig. 56 where the attenuation is greater for 20 miles of the same cable unloaded.

Lag of the Received Current in a Submarine Telephone Cable

A reference to Fig. 57 will show that the received current curve is displaced with reference to the sent curve, *i.e.* the consonant and vowel portions are not in the same vertical line. In this case the cable length of 96 nauts in combination with the inductance and capacity probably reduced the velocity of propagation sufficiently to cause the received current to lag behind the sent current. This suggests a possibility of measuring the velocity of propagation by the oscillograph. The necessary details are not, however, known in this case. The displacement is, however, what might be expected from the calculated velocity and average frequency of voice records such as that shown in Fig. 57.

93. Comparison of Speech Records taken by Different Means.—In order to permit of a comparison between the records taken by

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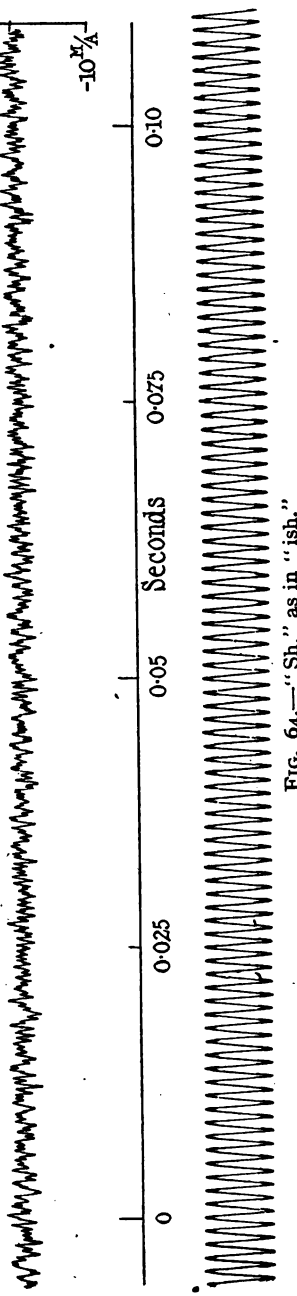

FIG. 64.—"Sh," as in "ish."

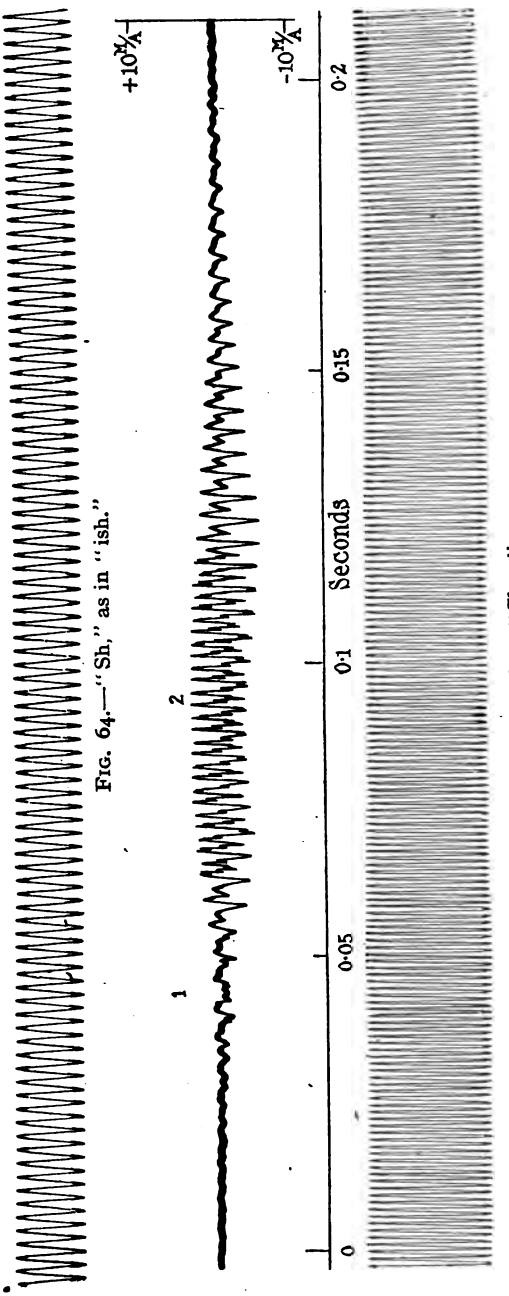

FIG. 65.—"Zhee."

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the method already described and given in this chapter, and other records taken by altogether different means, a few records have been taken from two independent sources.

Fig. 66 shows records of *p* and *b* followed by the letter *a*. These records are given in *Nature* for October 4th, 1917. They are from an abridged account of a discourse delivered at the Royal Institution on February 9th, 1917, by Mr. Daniel Jones, entitled "Experimental Phonetics and its Utility to the Linguist." The records were taken by means of a kymograph described in *Nature* in the issue of June 9th, 1917. If the *p* in Fig. 66 is compared with Fig. 52 it will be seen that in each case we have a straight line indicating the unvoiced part preceding the explosive consonant, which latter also appears in both figures, the vowels which follow in the two cases are different and cannot therefore be compared. The lower *b* in Fig. 66 may be compared with the record of the voiced part and the explosive

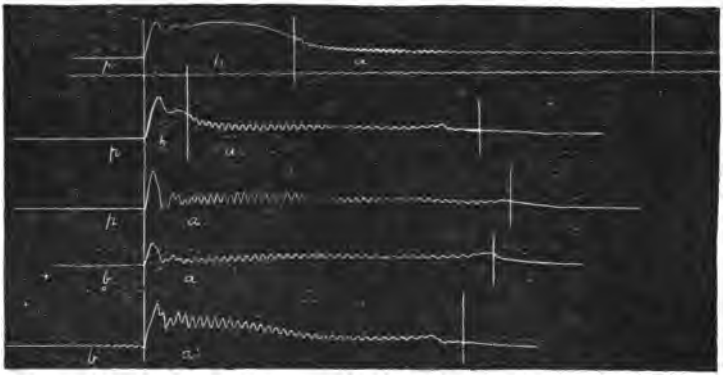

FIG. 66.—Reading from the top downwards—mouth—tracings of: fully aspirated *p*; partially aspirated *p*; unaspirated *p*; unvoiced *b*; and fully voiced *b*; each followed by the vowel *a*.

consonant in *b* as given in Fig. 55. The explosive is much greater in relative amplitude in Fig. 66. The comparison of the *p* and *b* may be said to show the same general facts, *i.e.* that in the *p* we have an unvoiced part and an explosive recorded in each case. In the case of the *b*, as was to be expected, the voiced part appears. The details in the two cases compared are not, however, very similar.

The records of *e*, also *f* and *s*, in Figs. 67 to 69 are taken from a paper on "The True Nature of Speech," by Mr. John B. Flowers, read before the American Institution of Electrical Engineers, New York, February 9th, 1916. In this case the records were made by an Einthoven string-galvanometer, and the records are those of whispered speech. Fig. 67 should be compared with Fig. 47 and other records containing *e*. The letter *f* in Fig. 60 has a general resemblance to the *f* in Fig. 68, and the letter *s* in Fig. 69 has many of the characteristics of the same letter in 62. The records are given for comparison, so that the reader may be in a position to

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judge for himself as to the degree of similarity, taking into account the difference of speakers and recording apparatus.

94. Quality of Speech in Different Kinds of Telephone Lines.—It is well known to telephone engineers who make comparative speaking tests on circuits of the various types referred to, that the character of the speech received on such lines varies considerably in clearness of articulation and pitch of sound. The speech received at the end of an electrically long unloaded submarine cable is relatively low in pitch and "drummy" owing to the fact that the upper harmonics have been more attenuated than the lower ones. Notwithstanding

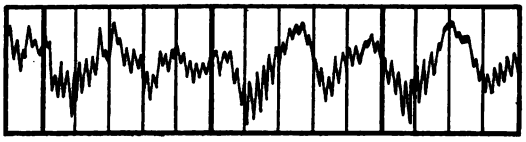

FIG. 67.—*e*, as in "me."

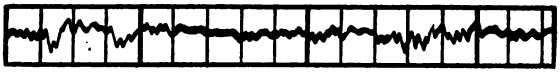

FIG. 68.—*f*, as in "fee."

FIG. 69.—*s*, as in "say."

this, speech on commercial subjects can be conducted over such cables. A long loaded circuit is higher in pitch and has a somewhat metallic ring, although the articulation is designed to be sufficiently clear. In an unloaded aerial line the speech is more natural, and there is very little distortion (see Chap. X.). It is clear from the foregoing observations that we may have a large variation from the conditions of ideal speech and still be in a position to conduct fairly satisfactory telephonic communication over long lines.

95. The Frequency of Current Speech Waves.—A glance at the various oscillograms here given will show that the composition of the speech waves varies considerably in different sounds. The fundamental tone of the vowel sounds varies generally between 200 and 300 periods per second, whilst the harmonics of the vowels in some cases visibly exceed 2000 vibrations per second. The highest vowel sound recorded appears to be the letter *e*. The consonants, except the explosives, have generally a less amplitude than the vowels, and they vary widely in the frequency of their fundamental tone. Now in order that speech may be perfectly reproduced, it is necessary that all these wave forms should be reproduced in the telephone receiver with sufficient amplitude to

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produce their corresponding audible sound, and that too with the same relative amplitudes as those impressed on the circuit at the beginning of the line; this is, moreover, on the assumption that the impressed wave at the beginning of the line correctly represents the wave impressed on the telephone transmitter diaphragm.

96. Remarks on Experiments made to determine the Mean Frequency of Speech.—In view of the widely varying and rapidly changing voice waves exhibited by the oscillograms here given, it is a matter of great difficulty to find the mean frequency of speech. If we consider the receiving end of a long telephone circuit and the attenuation to which speech waves are subjected in such a circuit, it is evident that the original relative amplitude of the various speech sounds as shown on the oscillograms, plays as important a rôle as the frequency in determining the possibility of satisfactory speech. Obviously if the received amplitude of a wave is too feeble to permit of its audible reproduction in the receiver owing to its initial small amplitude, even a relatively low frequency may cause difficulty, notwithstanding its small attenuation, whilst a sound of greater amplitude and higher frequency may be heard. A number of investigators have, however, made experiments with the object of finding the mean frequency of the human voice in telephony. Dr. Hayes reported experiments in the *Transactions* of the St. Louis International Electrical Congress (1904), Section G, pages 648–649. A curve was given showing the relative loudness of sound at various points along a telephone line as observed on a telephone receiver; from this Dr. Kennelly computed the equivalent frequency as 803 cycles per second from the known law of telephonic attenuation, and its dependence on frequency. M. Devaux Charbonnel, in "Étude sur les Lignes Téléphoniques; École professionnelle supérieure des Postes et Télégraphes," Paris, 1909, page 10, found the indispensable frequencies of the voice to lie between 800 and 1200 periods per second; the method adopted was to compare an ohmic resistance with an inductive resistance in a telephone circuit and to find when the two had the same impedance effect on speech; knowing the value of the ohmic resistance, the frequency was deduced from the impedance of an inductive resistance, this being a function of the frequency; this determines the upper limit of frequency. A capacity was similarly employed to find the lower limit. Prof. Breisig by an analogous method; see "Mitteilungen aus dem Telegraphen—Versuchsamt des Reichs-Postamts," vol v., page 98; Julius Springer, Berlin, found the mean frequency to be approximately 800 cycles per second. Experiments made in the Research Section of the Engineer in Chief's Office, Post Office, to find the upper and lower limits of frequency necessary for intelligible speech by the use of filters which shut out all frequencies above and below any desired number, led to the conclusion that the frequencies between 500 and 1600 are necessary for clear conversation. It is interesting to note that with all frequencies above 1000 cycles per second cut off, or with all frequencies below 1000 cut off, the limit of conversation was reached.

The foregoing review of some of the available evidence on the subject will enable the reader to appreciate the reasons for the adoption of a mean frequency not less than 800 cycles per second.

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This, however, in the author's opinion should not be too rigidly interpreted in view of the following facts. The different sounds composing speech are of widely differing frequencies and amplitudes, and unequal attenuation results as they progress along a given line. Now, as the upper overtones are more attenuated than the lower ones, depending on their frequency, they become extinguished before the lower ones if the amplitudes are equal, and it follows that the mean effective frequency of speech must gradually become less as the waves progress along the line and the circuit increases in length. (For evidence of this see Mr. C. E. Hay's remarks in the discussion on a paper by the author on "Aerial Lines and their Electrical Constants," read before the Institution of Post Office Electrical Engineers in January, 1914.) So that it is difficult to see how any fixed frequency can hold good. The following observations bear on this point.

In making tests of the comparative transmission efficiency of two uniform transmission circuits by means of a standard cable test, which is a method that has been largely used, it does not necessarily follow if equality of speech is obtained with given lengths of the two circuits (see Chap. XI.), and if the same ratio of length is obtained by calculation at some one frequency, that therefore the real mean frequency of speech is that taken in the calculation, as is sometimes assumed. Consider, for example, the case of two high resistance cables having the same electrostatic capacity per mile, and having resistances R and R_1 per mile. Then we have by (57), Chap. VIII.,

$$\frac{\beta}{\beta_1} = \sqrt{\frac{\frac{1}{2}w\mathrm{CR}}{\frac{1}{2}w\mathrm{CR}_1}} = \sqrt{\frac{\mathrm{R}}{\mathrm{R}_1}} \quad . \quad . \quad . \quad . \quad (1)$$

where $w = 2\pi f$.

It will be seen by inspection of the ratio of β to β_1 that the frequency cancels out, and that the same ratio is obtained for any one frequency. In actual practice, of course, the voice has many frequencies, but with the same type of cable the overtones become progressively extinguished with length in the same way by (1), so that the frequency does not affect the result. If, however, the standard cable is compared by a speech test with an unloaded aerial circuit (see Chap. X., Table II., and note that formula (48), Chap. VIII., applies to open lines), the less attenuation of the overtones in the aerial line would result in an increase of the length of line giving the same received current as the high resistance unloaded line, as compared with the case where the aerial line suffered the same attenuation of overtones as the cable. The effect of the smaller attenuation of the overtones in the aerial line would in fact make it appear that the mean frequency of speech, as judged by such a test, was higher than is the case in reality.

To illustrate this point, suppose that the standard cable has an attenuation constant β per unit length, and that the aerial circuit dealt with in Table II., Chap. X., has an attenuation constant β_1 per unit length. The ratio $\frac{\beta}{\beta_1}$ for three different frequencies would be as follows;—

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$2\pi f$	$\frac{\beta}{\beta_1}$	Ratio.	$2\pi f$	$\frac{\beta}{\beta_1}$	Ratio.	$2\pi f$	$\frac{\beta}{\beta_1}$	Ratio
3000	$\frac{0{\cdot}0833}{0{\cdot}00549}$	$=15{\cdot}2$	5000	$\frac{0{\cdot}1062}{0{\cdot}00574}$	$=18{\cdot}5$	7000	$\frac{0{\cdot}1241}{0{\cdot}00583}$	$=21{\cdot}47$

It will be seen that as the frequency increases the ratio $\frac{\beta}{\beta_1}$ increases, and this means that the ratio of length of aerial line to unit length of standard cable for the same received current also increases with frequency. Conversely a high ratio of length as determined by a standard cable test might lead to an assumption of a relatively high mean frequency, whereas the relatively high ratio might be due in part to the greater attenuation of overtones in the cable than in the aerial line. In any case the frequency cannot be directly deduced from the ratio $\frac{\beta}{\beta_1}$, because β varies approximately as the square root of the frequency, whilst β_1 is nearly constant with change of frequency.

Whatever may be the explanation, however, there is no doubt that the result of comparative speech tests on transmission lines, made by means of the standard cable, agree fairly well with a frequency assumed to be not less than 800 cycles per second.

97. Conclusions.—(*a*) Telephonic speech waves may be regarded in the great majority of cases as having attained the steady state.

(*b*) The question of the mean frequency of speech involves many complex factors, some of which do not yet appear to have been fully investigated.

(*c*) Speech tones in the neighbourhood of 800 cycles per second and for a considerable range of frequency on both sides of it are of great importance in telephony.

(*d*) For practical purposes, sine wave calculations based on a mean frequency of 800 cycles per second, may be used for the calculation of the results to be expected in speech tests on telephonic lines.

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CHAPTER VI

THE APPLICATION OF ALTERNATING CURRENTS TO TRANSMISSION LINES IN TELEPHONY

98. Introductory Remarks.—Although the transmission formulæ developed for the direct current, and alternating current cases, are the same (compare Appendix I. and Chap. III.) there is a marked difference in the calculation of the formulæ in the two cases. The direct current case is, in fact, only a particular case of the more general one which deals with alternating currents. The detailed application of the latter, which involves vector quantities, is much more involved than the former, which only requires the use of scalar quantities.

It must now be assumed that the reader is sufficiently familiar with Chapter I., including the portion entitled "Simple Harmonic Motion and the Sine Function," to follow the further developments and applications in this chapter.

99. Phase with Reference to the Sending End of the Line.—The propagation of a current of sine wave form, and its attenuation in an infinite line, are illustrated in Fig. 7. A series of exactly similar sine waves are generated by the source of energy joined to the line, and these are transmitted along the circuit. The current at the sending end is shown at the moment when it starts from zero, and it is assumed that the current has reached the steady state. If a point on the line be taken where the wave has its maximum amplitude at the same instant as the current at the sending end of the line has the value 0, say, for example, the point marked 9 on the horizontal line of Fig. 7, the phase at the latter point is $\frac{\pi}{2}$ radians, or 90 degrees different from that at the sending end. When the maximum amplitude is attained at the sending end, the amplitude of the current at the point 9 is at its minimum, *i.e.* zero; but the difference of phase between the sending-end and point 9 is still 90 degrees, and so long as the velocity of propagation and the impressed sine wave form and frequency remain constant, the phase difference will remain constant. The same reasoning applies to the phase difference at any other point on the line, and it may be positive or negative. It is evident from the figure that there may be a number of points on the line with the same phase difference with reference to the current at the beginning of the line. If a point on the line is considered by itself, however, without reference to any other point (say the case of point 9 in Fig. 7), the current in its propagation past that point passes through all possible instantaneous values f times per second, in this case $f = 750$.

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100. Amplitude and Vector.—So long as a fixed voltage is impressed at the sending end, and the circuit conditions remain constant, the maximum amplitude at a given point will be unaltered. In practice it is preferable to consider the current or voltage at any point on a transmission line in terms of its maximum amplitude and phase angle, as explained in Appendix I., in preference to the instantaneous value of the current or volts. In Figs. 1 and 6, Chap. I., A represents a maximum amplitude which is constant whatever be the angle ϕ or $\phi + wt$, and by para. 29, Chap. I., it follows that any voltage or current on a line may be represented by a vector, and in Appendix I. (20) an application of this equation is given. The vector quantity of volts or current thus defined has a constant modulus at any fixed point on a transmission line, no matter what the phase angle may be at any given instant, and therefore the ratio of two vectors at any two points represents the ratio of the maximum volts or current at those points, as the case may be.

101. Propagation of Volts and Current in Infinite Line.—The combined attenuation and change of phase per unit length of the volts and current in an infinite line is given by the complex attenuation constant or propagation constant, and this is by Appendixes I. and II.

$$\gamma = \beta + j\alpha = \sqrt{(R + jwL)(G + jwC)} \quad . \quad . \quad . \quad (1)$$

The current entering the infinite line is determined by the characteristic impedance Z_0, and by Appendixes I. and II.

$$Z_0 = \sqrt{\frac{R + jwL}{G + jwC}} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The current I_1 at the beginning of the infinite line is $\frac{E_1}{Z_0} = I_1$, and the current at any point l on an infinite line is by (2), Chap. II.

$$I_2 = \frac{E_1}{Z_0} e^{-\gamma l} = \frac{E_1}{Z_0} e^{-\theta} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly the volts at any point l are by (1), Chap. II.

$$E_2 = E_1 e^{-\gamma l}$$

If E_2 is divided by I_2 we obtain $\frac{E_2}{I_2} = Z_0$

Now Z_0 is a constant, and since $\frac{E_2}{I_2}$ is a constant quantity at any point of the line, the volts must, therefore, be attenuated in accordance with exactly the same law as the current, and further, as the constant Z_0 is generally a vector with an angle, the volts and current have everywhere the same constant phase difference. These relations only hold for an infinite line or for a finite line closed at the receiving end by an impedance Z_0 equal to the characteristic impedance of the line, ω is assumed to be constant.

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The current and voltage at intermediate points on finite lines follow more complex laws, and these will now be investigated.

102. Current at any Point x in a Uniform Transmission Line of any Length, Short Circuited at the Receiving End.—In Figure 70

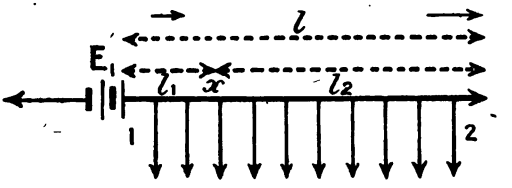

FIG. 70.

l is the total length of the circuit from 1 to 2;
l_1 is the variable distance of x from the sending end; and
$(l - l_1) = l_2$ is the variable distance of x from the receiving end 2.

If γ is the propagation constant, the following relations will hold:

$$\gamma l = \text{the line angle } \theta$$
$$\gamma l_1 = \quad ,, \quad ,, \quad \theta_1$$
$$\gamma(l - l_1) = \gamma l_2 = \quad ,, \quad ,, \quad \theta_2$$

The formula for the current Ix at x is deduced as follows:—

By (17), Chap. III., $$I_2 = \frac{E_1}{Z_0 \sinh \theta + Za \cosh \theta}$$

where Za is any terminal impedance. If Za be substituted by a transmission line having a sending-end impedance Za, the combination in (17), Chap. III., will be unaltered as regards its sending-end impedance, as will be further evident from a reference to formula (15), Chap. III., for such an impedance. Now Za is unrestricted as to magnitude, and it may be replaced by any impedance $Za = Z_1 = Z_0 \tanh \theta_2$. See (6), Chap. III. If, then, a line be substituted for Za in Fig. 16, and this line has the same characteristic impedance Z_0 as the line l_1, Fig. 70, the junction of the two lines will be represented by x, Fig. 70, and the length of line added will be that shown as l_2 in the same figure.

Formula (17), Chap. III., is now modified as follows:—

$$Ix = \frac{E_1}{Z_0 \sinh \theta_1 + Z_0 \tanh \theta_2 \cosh \theta_1}$$
$$= \frac{E_1}{Z_0 (\sinh \theta_1 + \tanh \theta_2 \cosh \theta_1)} \quad (4)$$

Where θ_1 and θ_2 have the meaning already explained; (4) may be written

$$Ix = \frac{E_1 \cosh \theta_2}{Z_0 \sinh \theta_1 \cosh \theta_2 + Z_0 \sinh \theta_2 \cosh \theta_1}$$
$$= \frac{E_1 \cosh \theta_2}{Z_0 \sinh (\theta_1 + \theta_2)} = \frac{E_1 \cosh \theta_2}{Z_0 \sinh \theta} \quad (5)$$

[See (42), Chap. I.]

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As an application of (4) the following illustration will serve :—

Since $(l - l_1) = l_2$, then if $l_1 = \frac{l}{2}$, x is at the middle point of l, and therefore $\gamma l_1 = \gamma l_2$, or $\theta_1 = \theta_2 = \frac{\theta}{2}$. In that case (4) becomes

$$I_x = \frac{E_1}{Z_0\left(\sinh \frac{\theta}{2} + \tanh \frac{\theta}{2} \cosh \frac{\theta}{2}\right)} = \frac{E_1}{2Z_0 \sinh \frac{\theta}{2}} \quad . \quad (6)$$

This is the current at the middle point of a line closed at the distant end. If the line had been closed direct at the middle point the current at that point would be

$$I_x = \frac{E_1}{Z_0 \sinh \frac{\theta}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

by (10), Chap. III., seeing that the line angle is now $\frac{\theta}{2}$.

It follows from (6) and (7) that the current at the middle point of a line which is earthed at the distant receiving end is half that at the same (middle) point when the line is earthed direct at that point.

103. Current at any Point x in a uniform finite Transmission Line open at the Receiving End.—The mode of procedure applied in the preceding case applies to this case also, and Fig. 70 illustrates the conditions if the receiving end is disconnected. In place of Za in (17), Chap. III., $Z_0 \coth \theta$, which is the formula for the sending-end impedance of a line disconnected at the distant end, is now substituted [see (5), Chap. III.]. The required current I_x is then

$$I_x = \frac{E_1}{Z_0 \sinh \theta_1 + Z_0 \coth \theta_2 \cosh \theta_1} \quad . \quad . \quad (8)$$

Multiply and divide (8) by $\sinh \theta_2$, and we obtain

$$I_x = \frac{E_1 \sinh \theta_2}{Z_0(\sinh \theta_1 \sinh \theta_2 + \cosh \theta_1 \cosh \theta_2)} = \frac{E_1 \sinh \theta_2}{Z_0 \cosh (\theta_1 + \theta_2)}$$

whence

$$I_x = \frac{E_1 \sinh \theta_2}{Z_0 \cosh \theta} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

104. Voltage at any Point in a uniform Transmission Line of any Length closed at the Receiving End.—Since $E = IZ$, the voltage at any point may be found by multiplying (5), which is the current at any point x when the receiving end is closed, by $Z_0 \tanh \theta_2$, which is the impedance measured from x with the sending end closed. We thus obtain

$$E_x = \frac{E_1 \cosh \theta_2}{Z_0 \sinh \theta} \times Z_0 \tanh \theta_2 = \frac{E_1 \sinh \theta_2}{\sinh \theta} \quad . \quad (10)$$

105. Voltage at any Point on a Uniform Transmission Line of any Length open at the Receiving End.—The required result is readily

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found by the same method as that adopted in the preceding case. We then obtain by (9), and (5), Chap. III.,

$$E_x = I_x Z_x = \frac{E_1 \sinh \theta_2}{Z_0 \cosh \theta} \times Z_0 \coth \theta_2 = \frac{E_1 \cosh \theta_2}{\cosh \theta} \quad (11)$$

106. Relation between Volts and Current in Finite Lines.—It is easily seen that the relation between volts and current in finite lines does not give the same constant phase relation as was found to be the case in an infinite line. The relation between the volts and current at any point x in a line closed at the receiving end is $\frac{E_x}{I_x} = Z_0 \tanh \theta_2$, which is the impedance measured to the receiving end from x. When the line is very long $\tanh \theta_2$ is constant, and this is the uniform condition for the infinite line. If, however, the line is finite the phase difference and change of magnitude of $\frac{E_x}{I_x}$ is measured by the variations of $\tanh \theta_2$ where θ_2 is a complex angle. For small values of θ_2, $\tanh \theta_2$ changes rapidly as the angle changes; but as θ_2 increases in magnitude $\tanh \theta_2$ changes less rapidly, and after $\theta_2 = 1$ the change is slow (see Fig. 73).

107. Power at any Point of a Transmission Line of any Length closed at the Receiving End.—This is obtained by multiplying (5) by (10), these being the current and volts on such a line. We have

$$P_x = \frac{E_1 \cosh \theta_2}{Z_0 \sinh \theta} \times \frac{E_1 \sinh \theta_2}{\sinh \theta} = \frac{E_1^2 \cosh \theta_2 \sinh \theta_2}{Z_0 \sinh^2 \theta} \quad (12)$$

Since $\sinh 2\theta_2 = 2 \sinh \theta_2 \cosh \theta_2$ (12) becomes

$$P_x = \frac{E_1^2 \sinh 2\theta_2}{2Z_0 \sinh^2 \theta} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

108. Power at any Point x in a Transmission Line of any Length, open at the Receiving End.—We have again $P_x = E_x I_x$, and therefore multiplying the appropriate equations for volts and current we obtain from (9) and (11)

$$P_x = \frac{E_1 \sinh \theta_2}{Z_0 \cosh \theta} \times \frac{E_1 \cosh \theta_2}{\cosh \theta} = \frac{E_1^2 \sinh \theta_2 \cosh \theta_2}{Z_0 \cosh^2 \theta} \quad (14)$$

As in the case of (12) the numerator of (14) may be substituted by $\frac{E_1^2 \sinh 2\theta_2}{2}$, and (14) then becomes

$$P_x = \frac{E_1^2 \sinh 2\theta_2}{2Z_0 \cosh^2 \theta}$$

(4) to (14) refer to D.C. or A.C. cases, but it was convenient to develop the formulæ here.

109. Examples in the working of Alternating Current Problems.—It has already been pointed out that the only difference between alternating current and direct current transmission formulæ is that the former involve vector quantities, whereas the latter do not. The difference is, however, one that involves considerable complexity in the alternating current case, and in order to illustrate

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the method of solving such problems some worked examples will now be given.

Example 1.—Calculate the characteristic impedance and the attenuation constant for standard cable, which has the following constants per loop mile :—

Resistance (R) = 88 ohms
Inductance (L) = 0·001 henry
Capacity (C) = 0·054 microfarad
Leakance (G) = 1 micromho
Assume $\omega = 2\pi f = 5000$.

This example furnishes the method for calculating the transmission data given in Table II., Chap. VIII. It should be noted that although (36), Appendix I., gives a formula for the direct calculation of the attenuation constant, this formula is not used in this case, as it will be found simpler to calculate the real part of the propagation constant γ by (1) which gives exactly the same result as (36), Appendix I.

We have

$$Z_0 = \sqrt{\frac{R + j\omega l}{G + j\omega C}} \quad \text{[by (2)]}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \text{[by (1)]}$$

From (18), Chap. I., we deduce

$$R + j\omega L = \sqrt{R^2 + \omega^2 L^2} \angle \tan^{-1} \frac{\omega L}{R} = A \angle \phi = 88{\cdot}14 \angle 3^\circ\ 15'$$

$$G + j\omega C = \sqrt{G^2 + \omega^2 C^2} \angle \tan^{-1} \frac{\omega C}{G} = A_1 \angle \phi_1 = 270 \times 10^{-6} \angle 89^\circ\ 47'$$

Hence $Z_0^2 = \dfrac{A \angle \phi}{A_1 \angle \phi_1} = \dfrac{A}{A_1} \angle \phi - \phi_1 = \dfrac{88{\cdot}14}{270 \times 10^{-6}} \angle 86^\circ\ 32'$

by (24) Chap. I. And

$$Z_0 = \sqrt{\frac{A}{A_1}} \angle \frac{\phi - \phi_1}{2} = \sqrt{\frac{88{\cdot}14}{270 \times 10^{-6}}} \angle \frac{86^\circ\ 32'}{2} = 571{\cdot}4 \angle 43^\circ\ 16'$$

by (26), Chap. I. Also

$$\gamma = \sqrt{AA_1} \angle (\phi + \phi_1) \text{ see (23), Chap. I.} = \sqrt{88{\cdot}4 \times 270 \times 10^{-6}} \angle 93^\circ 2'$$

$$= \sqrt{AA_1} \angle \frac{\phi + \phi_1}{2} \text{ see (26), Chap. I.} = 0{\cdot}1543 \angle 46^\circ\ 31'$$

$$= A_2 \angle \phi_2 = \beta + j\alpha = 0{\cdot}10616 + j\ 0{\cdot}11193$$

where $\alpha = A_2 \sin \phi_2$ and $\beta = A_2 \cos \phi_2$ (see Fig. 1, Chap. I.).

This example will serve for reference in future similar cases, it will not then be necessary to repeat the full details here given.

Example 2.—(*a*) A double wire underground circuit consisting of 70 lbs. Cu conductors, and having the constants given in Table I.,

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Chap. II., has 10 volts applied at the sending end. Assuming that the line is infinitely long, find the distance from the sending end and the value of the current at the point where $\beta l = 1$ and where $\beta l = 2$. What is the value of the characteristic impedance of this circuit if $\omega = 5000$?

$$R + j\omega L = 25{\cdot}14 + 5j = 25{\cdot}63 \underline{/ 11^\circ\ 15'}$$

$$G + j\omega C = 10^{-6} + 325 \times 10^{-6} j = 325 \times 10^{-6} \underline{/ 89^\circ\ 49'}$$

$$\therefore \gamma = \sqrt{25{\cdot}63 \times 325 \times 10^{-6}} \underline{/ 101^\circ\ 4'}$$

$$= 0{\cdot}0913 \underline{/ 50^\circ\ 32'} = 0{\cdot}0580 + j\,0{\cdot}0705$$

and

$$Z_0 = \sqrt{\frac{25{\cdot}63}{325 \times 10^{-6}}} \overline{\backslash 78^\circ\ 34'} = 280{\cdot}84 \overline{\backslash 39^\circ\ 17'}$$

If $\beta l = 1$, then

$$l = \frac{1}{0{\cdot}0580} = 17{\cdot}24 \text{ mls.}$$

hence

$$\alpha l = 1{\cdot}2145$$

Current at this point

$$= \frac{E_1 e^{-\gamma l}}{Z_0} \text{ (from (2), Chap. II.)}$$

$$= \frac{10 \times e^{-1} \times e^{-j\,1{\cdot}2145}}{280{\cdot}84 \overline{\backslash 39^\circ\ 17'}} = \frac{3{\cdot}6788 \overline{\backslash 69^\circ\ 35'}}{280{\cdot}84 \overline{\backslash 39^\circ\ 17'}}$$

$$= 13{\cdot}099 \times 10^{-3} \overline{\backslash 30^\circ\ 18'} \text{ amps.}$$

$$= 13{\cdot}099 \overline{\backslash 30^\circ\ 18'} \text{ milliamps.}$$

[See (19), Chap. I. and (24), Chap. I.]

If $\beta l = 2$, then

$$l = \frac{2}{0{\cdot}0580} = 34{\cdot}47 \text{ mls.}$$

hence

$$\alpha l = 2{\cdot}4291$$

Current at this point

$$= \frac{10 \times e^{-2} \times e^{-j\,2{\cdot}4291}}{280{\cdot}84 \overline{\backslash 39^\circ\ 17'}}$$

$$= \frac{1{\cdot}3533 \overline{\backslash 139^\circ\ 11'}}{280{\cdot}84 \overline{\backslash 39^\circ\ 17'}}$$

$$= 4{\cdot}819 \times 10^{-3} \overline{\backslash 99^\circ\ 54'} \text{ amps.}$$

$$= 4{\cdot}819 \overline{\backslash 99^\circ\ 54'} \text{ milliamps.}$$

Example 3.—Find the sending-end impedance and current, and also the current at the receiving end of a closed double wire 70 lbs. unloaded telephone cable circuit, which has an attenuation length

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such that $\beta l = 1$ when the impressed voltage is 10 volts, having given that $2\pi f = 5000$.

The characteristic impedance and propagation constant may be taken from Table II., Chap. VIII.

We have
$$Z_0 = 280{\cdot}84 \angle 39^\circ\ 17'$$
$$\alpha = 0{\cdot}070462$$
$$\beta = 0{\cdot}058016$$

$\theta = (\beta + j\alpha)l$, and it is given that $\beta l = 1$, $\therefore l = \frac{1}{\beta} = \frac{1}{{\cdot}058016} = 17{\cdot}24$

whence
$$\alpha l = 1{\cdot}2145 \text{ and } \theta = (1 + j1{\cdot}2145)$$
$$Z_1 = Z_0 \tanh \theta \text{ (by (6), Chap. III.)}$$
$$I_1 = \frac{E_1}{Z_0 \tanh \theta} \text{ (by (9), Chap. III.)}$$
$$I_2 = \frac{E_1}{Z_0 \sinh \theta} \text{ (by (10), Chap. III.)}$$

From (78) and (79), Chap. I., we deduce

$$\sinh(\beta + j\alpha) = \sinh\beta\cos\alpha + j\cosh\beta\sin\alpha$$
$$= \sinh 1 \cos 1{\cdot}2145 + j \cosh 1 \sin 1{\cdot}2145$$
$$= 1{\cdot}1752 \times 0{\cdot}3488 + j\ 1{\cdot}54308 \times 0{\cdot}9372$$
$$= a + jb = A\underline{/\phi} = (0{\cdot}4099 + j1{\cdot}4462) = 1{\cdot}5032\underline{/74^\circ\ 10'}$$

$$\cosh(\beta + j\alpha) = \cosh\beta\cos\alpha + j\sinh\beta\sin\alpha$$
$$= \cosh 1 \cos 1{\cdot}2145 + j \sinh 1 \sin 1{\cdot}2145$$
$$= 1{\cdot}54308 \times 0{\cdot}3488 + j\ 1{\cdot}1752 \times 0{\cdot}9372$$
$$= a_1 + jb_1 = A_1\underline{/\phi_1} = (0{\cdot}5382 + j1{\cdot}1014) = 1{\cdot}2258\underline{/63^\circ\ 57'}$$

$$\tanh\theta = \tanh(\beta + j\alpha) = \frac{\sinh(\beta + j\alpha)}{\cosh(\beta + j\alpha)} = \frac{1{\cdot}5032\underline{/74^\circ\ 10'}}{1{\cdot}2258\underline{/63^\circ\ 57'}}$$

whence
$$\frac{A\underline{/\phi}}{A_1\underline{/\phi_1}} = \frac{A}{A_1}\underline{/\phi - \phi_1} = 1{\cdot}2263\underline{/10^\circ\ 13'}$$

$$Z_1 = Z_0 \tanh\theta = 280{\cdot}84\angle 39^\circ\ 17' \times 1{\cdot}2263\underline{/10^\circ\ 13'}$$
$$= 280{\cdot}84 \times 1{\cdot}2263(\angle 39^\circ\ 17' + \underline{/10^\circ\ 13'})$$
$$= 344{\cdot}4\angle 29^\circ\ 4' \text{ vector ohms}$$

$$I_1 = \frac{E_1}{Z_0 \tanh\theta} = \frac{10}{344{\cdot}4\angle 29^\circ\ 4'} = \underline{29{\cdot}04 \times 10^{-3}}\ \underline{/29^\circ\ 4'} \text{ amps.}$$

$$I_2 = \frac{E_1}{Z_0 \sinh\theta} = \frac{10}{280{\cdot}84\angle 39^\circ\ 17' \times 1{\cdot}5032\underline{/74^\circ\ 10'}}$$
$$= \frac{E_1}{A_2 A(\underline{/\phi_2} + \underline{/\phi})} = \frac{E_1}{A_2 A}(\underline{/\phi_2} + \angle\phi)$$
$$= \underline{23{\cdot}69 \times 10^{-3}}\angle 34^\circ\ 53' \text{ amps.}$$

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By means of the data given in the foregoing example, the impedance and current at the sending end, and the receiving-end impedance and voltage, with the line open, may readily be found by the aid of formulæ (5), (8), and (13), Chap. III., and it may also be verified that $\frac{I_2}{I_1} = \frac{1}{\cosh \theta}$ as given by (12), Chap. III. This latter equation is frequently useful, and it is easily seen that the relation in question holds. We have from Example (3)

$$\frac{I_2}{I_1} = \frac{23{\cdot}69 \times 10^{-3} \angle 34^\circ\ 53'}{29{\cdot}04 \times 10^{-3} \angle 29^\circ\ 4'} = 0{\cdot}8158 \angle 63^\circ\ 57'$$

$$= \frac{1}{1{\cdot}2258 \angle 63^\circ\ 57'} = \frac{1}{\cosh (1 + j\,1{\cdot}2145)} = \frac{1}{\cosh \theta}$$

Example 4.—The circuit described in question 3 has now a set of telephone apparatus joined in series at each end of the line as represented in Fig. 18, Chap. III. The impedance of the two sets is the same, viz. $425 \angle 45^\circ$. Find the received current I_2 by means of formula (23), Chap. III., $E_1 = 10$ volts.

As in the preceding example $Z_0 = 280{\cdot}8 \angle 39^\circ\ 17'$

$$\therefore\ Z_0^2 = (280{\cdot}8 \angle 39^\circ\ 17')^2 = 78849 \angle 78^\circ\ 34' = 15630 - j\,77284$$

$$Za^2 = (425 \angle 45^\circ\ 0')^2 = 180625 \angle 90^\circ\ 0' = 0 + j\,180625$$

$$Z_0^2 + Za^2 = 15630 + j\,103341 = 104510 \angle 81^\circ\ 24'$$

$$\frac{Z_0^2 + Za^2}{Z_0} = \frac{104510 \angle 81^\circ 24'}{280{\cdot}8 \angle 39^\circ\ 17'} = 372{\cdot}2 \angle 120^\circ\ 41'$$

$$I_2 = \frac{E_1}{\left(\frac{Z_0^2 + Za^2}{Z_0}\right) \sinh \theta + 2Za \cosh \theta}$$

The values of $\sinh \theta$ and $\cosh \theta$ are the same as in Example 3, we therefore have

$$I_2 = \frac{10}{372{\cdot}2 \angle 120^\circ\ 41' \times 1{\cdot}5032 \angle 74^\circ\ 10' + 2 \times 425 \angle 45^\circ \times 1{\cdot}2258 \angle 63^\circ\ 57'}$$

$$= \frac{10}{559{\cdot}5 \angle 194^\circ\ 51' + 1041{\cdot}8 \angle 108^\circ\ 57'}$$

$$= \frac{10}{-540{\cdot}8 - j\,143{\cdot}4 - 338{\cdot}3 + j\,9875{\cdot}4}$$

It should be noticed that as $194^\circ\ 51'$ is an angle in the third quadrant the horizontal part of the vector is from right to left and the vertical component is downward, and therefore both are negative. $108^\circ\ 57'$ is in the second quadrant, and in this case the horizontal component only is negative.

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$$= \frac{10}{-879{\cdot}1 + j\,842{\cdot}0}$$

$$= \frac{10}{1217\,/133^\circ\,46'}$$

$$= 8{\cdot}22 \times 10^{-3}\,\backslash 133^\circ\,46' \text{ amps.}$$

A comparison of the current in this case with that obtained in Example 3 shows that the vector current has been reduced from 23·69 to 8·22 milliampères with a large change of angle, by the addition of the necessary apparatus at the two ends of the circuit. The maximum power in the sending and receiving apparatus is, however, obtained when the impedance of the apparatus is equal to that of the characteristic impedance of the line. (See Chapter VII.)

Calculation of Equivalent T *and* π *Circuits.*—In constructing such circuits for practical telephone use the double T and π circuits are much more useful than the single T and π form, owing to the fact that the former are symmetrical when joined to double-wire telephone circuits, whereas the latter are not. The difference between the two will be seen by a comparison of Figs. 9 and 11 and 20 and 21. Examples of T and single π circuits will now be given; but the data suffice for both types.

It is necessary in the alternating current case to take frequency into account, and the frequency selected is such that $2\pi f = 5000$ for reasons given in Chap. V. All the constants given are "effective" values at that frequency. The proofs of the formulæ are given in Appendixes III. and IV.

Example 5.—Construct T and π circuits to be equivalent at $\omega = 5000$ to 20 and 46 miles of standard cable. Which of the two arrangements is the more suitable for use in these cases, and why?

Now $\theta = (0{\cdot}10616 + j\,0{\cdot}11193)l$
and $Z_0 = 571{\cdot}4\,\backslash 43^\circ\,16'$ } See Example 1.

If $l = 20$ mls.

then $\theta = 2{\cdot}1232 + j\,2{\cdot}2386$

$$\tanh\frac{\theta}{2} = \tanh(1{\cdot}0616 + j\,1{\cdot}1193) \text{ [(80), Chap. I.]}$$

$$= \frac{\tanh 1{\cdot}0616 + j\tan 1{\cdot}1193}{1 + j\tanh 1{\cdot}0616\tan 1{\cdot}1193}$$

$$= \frac{{\cdot}7864 + j\,2{\cdot}0625}{1 + j\,1{\cdot}6220}$$

$$= \frac{2{\cdot}2073\,/\,69^\circ\,7'}{1{\cdot}9056\,/\,58^\circ\,21'}$$

$$= 1{\cdot}1583\,/\,10^\circ\,46'$$

$$\sinh\theta = \sinh(2{\cdot}1232 + j\,2{\cdot}2386) \text{ [by (78), Chap. I.]}$$

$$= \sinh 2{\cdot}1232\cos 2{\cdot}2386 + j\cosh 2{\cdot}1232\sin 2{\cdot}2386$$

$$= 4{\cdot}1182 \times (-{\cdot}6163) + j\,4{\cdot}2379 \times {\cdot}7875$$

$$= -2{\cdot}5381 + j\,3{\cdot}3374$$

$$= 4{\cdot}193\,/\,127^\circ\,15'.$$

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$$\therefore Z_0 \sinh\theta = 571{\cdot}4\angle 43^\circ 16' \times 4{\cdot}193\angle 127^\circ 15' = 2396\angle 83^\circ 59'$$

$$\text{and} \quad \frac{Z_0}{\tanh\frac{\theta}{2}} = \frac{571{\cdot}4\angle 43^\circ 16'}{1{\cdot}1583\angle 10^\circ 46'} = 493{\cdot}3\angle 54^\circ 2'$$

$$\text{also} \quad \frac{Z_0}{\sinh\theta} = \frac{571{\cdot}4\angle 43^\circ 16'}{4{\cdot}193\angle 127^\circ 15'} = 136{\cdot}2\angle 170^\circ 31'$$

$$\text{and } Z_0\tanh\frac{\theta}{2} = 571{\cdot}4\angle 43^\circ 16' \times 1{\cdot}1583\angle 10^\circ 46' = 661{\cdot}9\angle 32^\circ 30'$$

The equivalent circuits for 20 mls. of standard cable are therefore as shown in Figs. 71 and 72.

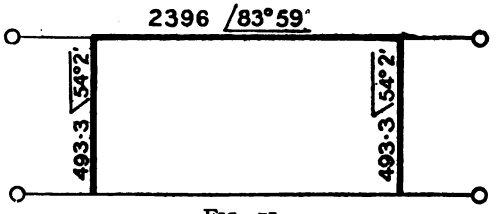

FIG. 71.

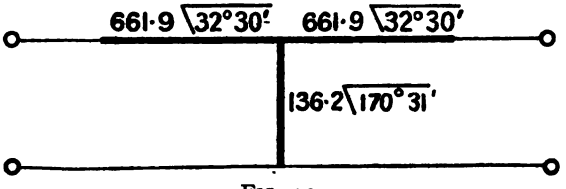

FIG. 72.

The π circuit is the more suitable because none of the impedances composing it have an angle greater than 90°, so that it can be made up without the aid of a transformer. The impedance $2396\angle 83^\circ 59'$ can be represented by a coil of 251·1 ohms in resistance and 476·5 mhys. in inductance; the impedance $493{\cdot}3\angle 54^\circ 2'$ by a resistance of 289·7 ohms in series with ·501 mfds.

If $l = 46$ mls.

then $\theta = 4{\cdot}8834 + j\,5{\cdot}1488$

$$\therefore \tanh\frac{\theta}{2} = \tanh(2{\cdot}4417 + j\,2{\cdot}5744)$$

$$= \frac{{\cdot}98497 - j\,{\cdot}63707}{1 - j\,{\cdot}98497 \times {\cdot}63707}$$

$$= \frac{1{\cdot}1731\angle 32^\circ 54'}{1 - j\,{\cdot}62750}$$

$$= \frac{1{\cdot}1731\angle 32^\circ 54'}{1{\cdot}1806\angle 32^\circ 7'} = {\cdot}9959\angle 0^\circ 47'$$

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$$\sinh\theta = \sinh(4{\cdot}8834 + j\,5{\cdot}1488)$$
$$= 66{\cdot}036 \times {\cdot}42262 - j\,66{\cdot}044 \times {\cdot}90631$$
$$= 27{\cdot}908 - j\,59{\cdot}856$$
$$= 66{\cdot}04\,\angle 65^\circ\,0'$$

$$\therefore Z_0 \sinh\theta = 571{\cdot}4\,\angle 43^\circ\,16' \times 66{\cdot}04\,\angle 65^\circ\,0' = 3774\,\angle 108^\circ\,16'$$

and $$\frac{Z_0}{\tanh\frac{\theta}{2}} = \frac{571{\cdot}4\,\angle 43^\circ\,16'}{{\cdot}9959\,\angle 0^\circ\,47'} = 573{\cdot}5\,\angle 42^\circ\,29'$$

$$Z_0 \tanh\frac{\theta}{2} = 571{\cdot}4\,\angle 43^\circ\,16' \times {\cdot}9959\,\angle 0^\circ\,47' = 569{\cdot}1\,\angle 44^\circ\,3'$$

$$\frac{Z_0}{\sinh\theta} = \frac{571{\cdot}4\,\angle 43^\circ\,16'}{66{\cdot}04\,\angle 65^\circ\,0'} = 8{\cdot}65\,/21^\circ\,44'$$

The T circuit is the more suitable in this case for the same reason as given before. The impedance $569{\cdot}1\,\angle 44^\circ\,3'$ can be represented by a resistance of 409 ohms in series with ·505 mfd.; the impedance $8{\cdot}65\,/21^\circ\,44'$ by a coil of 8·03 ohms in resistance and ·64 mhy. in inductance.

These examples show that there is not always freedom of choice between an equivalent T circuit and an equivalent π circuit if transformers are to be avoided. Generally, but not always, either the T or the π circuit permits of this being done. It should be carefully noted that these circuits as calculated are only strictly correct for a frequency such that $2\pi f = 5000$. In order that they may satisfactorily replace cables with uniformly distributed constants for speech tests, it is necessary to demonstrate by experiment that speech conducted on real cables and equivalent circuits gives similar results as regards attenuation and distortion of speech. It is also of importance that the theoretical constants found by calculation are such as may be reproduced in practical construction in a compact and economical manner.

Some experiments bearing on these points have been made in the Post Office Laboratory on a circuit similar to that shown in Fig. 72, and the results are indicated in the chapter on Alternating Current Measurements in connection with Fig. 128 in that chapter.

Example 6.—A 70-lbs. double wire circuit having the constants shown in Table II., Chap. VIII., is closed at the receiving end without apparatus. Show by a curve how the impedance varies as measured at the sending end of such a line when the circuit is gradually increased in length from $\beta l = 0$ to $\beta l = 4$, observations being continuously taken as the length is increased. By (6), Chap. III., the impedance of a closed circuit is $Z_0 \tanh\theta$ for any value of θ, the variable part of this is $\tanh\theta$ where θ is a complex angle. Fig. 73 shows the relation by means of a polar diagram. The curved line shows $\tanh\theta$ and the figures marked upon the curve from ·1 to 4 are the values of βl, and $\tanh\theta$ has then the value indicated by the curve. Example, when $\beta l = 1{\cdot}0$ the angle of $\tanh\theta$ is then a little over 10° and is positive. The corresponding vector is 1·226, as

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shown by the diagonal scale at the foot of the curve. The angle of 10° will be found on the right-hand edge of the graph, and it is opposite $\beta l = 1$. The precise meaning of the curve will be made clear by Example 3, which includes the calculation of tanh θ, and it will be seen that in the example taken the value is tanh $(1 + j\,1{\cdot}2145) = 1{\cdot}2263 \angle 10°\,13'$, which agrees with the curve.

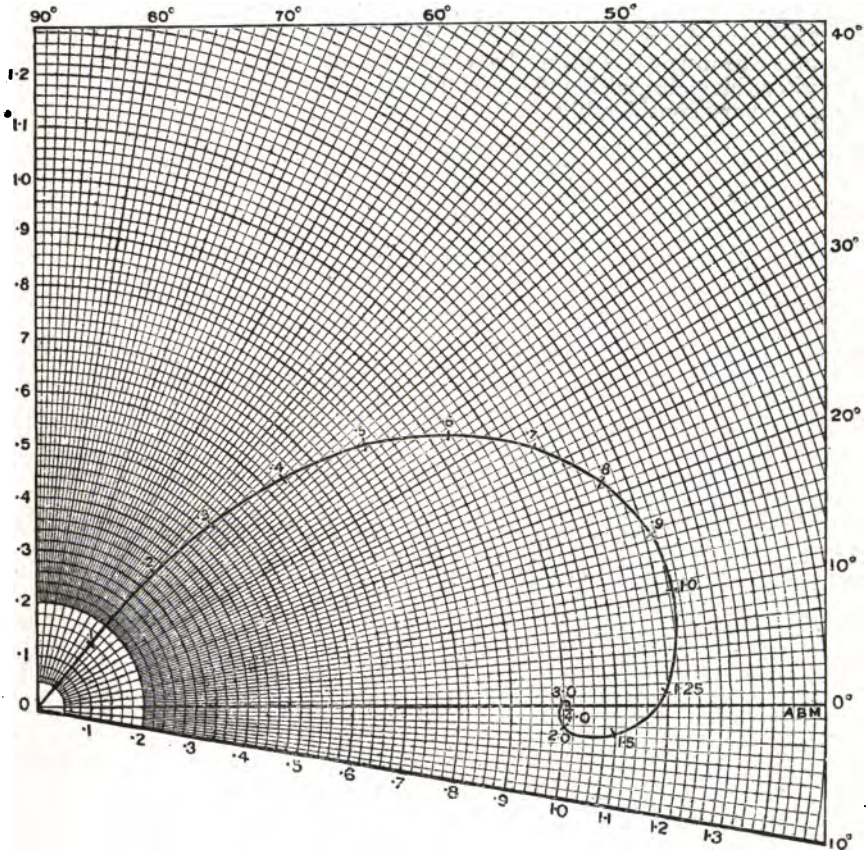

FIG. 73.—Polar Diagram, showing variation of tanh θ, for a 70-lbs. underground line of any length.

In order to obtain the value of the impedance, which is Z_0 tanh θ in this case, it will be observed that Z_0 is multiplied by $1{\cdot}2263 \angle 10°\,13'$, the value just found from the curve, and the example shows how to complete the calculation. In a similar manner the impedance of a closed 70 lbs. cable circuit for any line angle between 0 and 4 may be obtained by the aid of the curve. A study of the graph brings out some points of interest; for example, for small values of θ

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the value of tanh θ is nearly the same as θ, and it may be deduced from this that the impedance of a short line is practically $Z_0\theta$. The curve also shows that after $\beta l = 1{\cdot}25$, the change of the line angle θ is relatively small, the range of variation being about 4 degrees between the extreme limits. Also after $\beta l = 3$ the angle is nearly constant. We then have an electrically long line, and the curve beyond this point approximates closely to the requirements of an infinite line. In such a case tanh θ approximates to 1, and the curve shows the vector to be 1 and the angle 0, *i.e.* the impedance and the angle between the current and volts becomes constant after this point, although for small values of θ the curve shows that they vary considerably.

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CHAPTER VII

REFLECTION AND POWER IN TELEPHONE CIRCUITS

110. General Remarks.—The motion of waves in transmitting media, and their partial or total reflection when the nature of the medium changes, are familiar phenomena in Sound, Light, and other branches of Physics. Maxwell discovered that waves of light are electromagnetic waves, and Hertz showed that electromagnetic waves travel at the same speed as light, and are capable of being reflected and refracted.

Oliver Heaviside appears to have been the first to explain phenomena which occur at the boundary of unequal impedances in transmission lines, by the action of partial transmission and partial reflection of the electric waves at such points. (See "Electrical Papers," vol. ii. p. 131.) In accordance with this theory it will now be assumed that both volts and current are so reflected and transmitted. It will further be supposed that the harmonic variation and propagation of waves of volts and current act in an analogous way to other waves in media where their properties can be more readily demonstrated, such, for example, as water waves, and the vibrations of strings, and the reader is referred to treatises for fuller details. (See, for example, Franklin's "Electric Waves," and Prof. Fleming's paper read before the Royal Society entitled "An Experimental Method for the Production of Vibrations on Strings, Illustrating the Properties of Loaded or Unloaded Telephone Cables.") It will be shown that these assumptions of wave motion combined with elementary known electrical conditions at any point where reflection occurs in a transmission line, furnish the necessary data for the formation of reflection formulæ which may be used to develop transmission equations such as those already deduced in Chapter III. The conditions necessary for the establishment of transmission without reflection in a telephone line are also important in connection with the study of transmission losses.

When electric waves are propagated along a uniform transmission line of infinite length they travel with uniform speed and are continuously and uniformly attenuated, in accordance with formulæ (36) to (38), Appendix I. This in fact is the only case where reflection in a transmission line does not exist, if we except the case where the conditions of the infinite line are fulfilled by a finite circuit which is closed through an impedance equal to that of the infinite line. If the line changes in impedance, the velocity of propagation changes by (38), Appendix I., and this change is accompanied by reflection. Consider the lines represented in

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Fig. 74. We have two circuits in series, one being of finite, and the other of infinite length, having characteristic impedances Z_0 and Z'_0 respectively. The volts and current arriving at B from A in the first instance are supposed on the reflection theory to have exactly the same value as if the line were a uniform and infinite one, *i.e.* the volts are $E_1 = Ee^{-\theta}$ and the current $I_1 = \frac{E}{Z_0}e^{-\theta}$, where E is the impressed voltage at the sending end of the line, Z_0 the

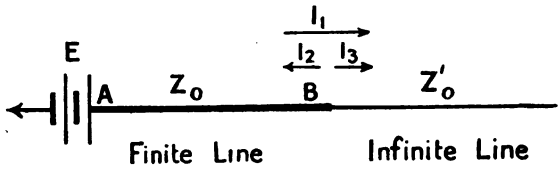

FIG. 74.

characteristic impedance of the line AB, and θ the line angle of that circuit. A portion of the volts and current thus defined are transmitted at B along Z'_0 and a part reflected from B to A, the magnitude of the transmitted and reflected portions depending on the relative impedance of Z_0 and Z'_0. If Z'_0 is infinitely great or zero the reflection is total, but the volts and current are differently affected in these two cases, as will shortly be shown. The waves reflected at B return to A with undiminished velocity and they are totally reflected at that point (if the generator has no resistance), and return to B, where the previous action at that point is repeated, and this process theoretically continues to infinity; each time that the current arrives at B the same process of partial transmission and reflection is repeated, the reflected waves as they run to and fro between A and B are continually attenuated in the same way as if the length AB were part of an infinite and uniform line.

111. Reflection at the Junction of Two Infinite Lines.—On account

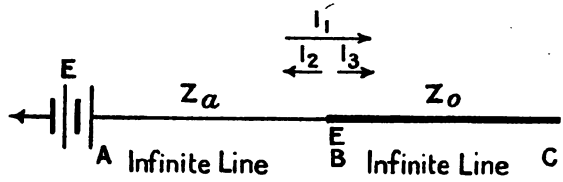

FIG. 75.

of the simplicity of the infinite line case it will be convenient to consider it first. Fig. 75 shows two lines of infinite length joined in series, Z_a and Z_0 being the characteristic impedances of those lines. The same result would be obtained as regards reflection, if the line AB were short and the generator had an impedance Z_a. We will suppose that the line Z_a remains constant in impedance, but that the line Z_0 may be changed for any other line of infinite length with a different characteristic impedance,

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which may have any impedance value Z_0 from zero to infinity. It will be convenient to consider the limiting values zero and infinity of Z_0 at this stage. In each case the volts and current reflected at B are respectively equal to the incident volts $Ee^{-\theta}$ at that point, and to the current $\frac{E}{Z_a}e^{-\theta}$. The results of this total reflection may be interpreted by the aid of the known electrical conditions at B when the line is disconnected or earthed at that point. The assumption of total reflection in these cases is in accordance with the physical analogies referred to, and also with well-known properties of electromagnetic waves, apart from other considerations; and it will be seen that the formulæ deduced on this supposition are in accordance with other formulæ deduced independently.

112. Reflection at the Disconnected End of a Transmission Line.—In Fig. 75 let the line Z_0 be disconnected at B. It is then known that no current exists at the point B, seeing that the impedance at the disconnection is infinite, but a resultant voltage E_3 is known to exist there, due to the action of the impressed volts at the sending end on the impedance at the receiving end, and E_3 is known to be of the same sign as the impressed volts. Let the volts $Ee^{-\theta}$ incident at B be called E_1 and be positive, let the reflected portion of E_1 be E_2, and the resultant of E_1 and E_2 be E_3. Similarly, let the current incident at B be I_1, the reflected current I_2, and the resultant value of I_1 and I_2 be I_3. Since we have total reflection we know that $E_1 = E_2$, and, as already stated, we know that the resultant voltage E_3 is not zero and is positive. It follows that E_2 is positive, for otherwise $E_1 - E_2 = 0$, which is not the case, therefore we must have

$$E_3 = E_1 + E_2 = 2E_1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As regards the current at the same point, since the resultant of the incident and reflected currents is known to be zero, and since total reflection exists we must have

$$I_3 = I_1 - I_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Formulæ (1) and (2) may be stated thus: At a point of disconnection, the incident voltage is equal to the reflected voltage and of the same sign; the incident current is equal to the reflected current and of opposite sign.

113. Reflection at the Short-circuited End of a Transmission Line.—In this case the resultant volts $E_3 = 0$, since the line is at zero potential at B. A current, however, exists at B due to the sending generator and the short-circuited or earthed line. In the case of both volts and current we have again total reflection. The value of E_3 is obtained thus

$$E_3 = E_1 - E_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This follows from the fact that $E_1 = E_2$, and E_3 is known to be 0.

The corresponding current equation is

$$I_3 = I_1 + I_2 = 2I_1 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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This follows from the fact that $I_1 = I_2$, and I_3 is greater than 0 and positive.

Formulæ (3) and (4) may be defined thus: At the short-circuited end of a transmission line, the incident voltage at that point is equal and of opposite sign to the reflected voltage; the incident current is equal to and of the same sign as the reflected current.

It should be noticed that in the cases of both disconnection and short circuit the reflected voltage is of opposite sign to the reflected current.

Combining equations (1) and (3) we have

$$E_3 = E_1 \pm E_2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When the voltage has these values the simultaneous value of the current is from (2) and (4)

$$I_3 = I_1 \mp I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The reversal of sign in the case of E_2 and I_2 in (5) and (6) indicates that E_2 is always of opposite sign to I_2.

114. Verification of Formulæ derived by the Reflection Process.—The results obtained are in accordance with formulæ already developed in Chap. III. To show that the volts would be doubled in a disconnected infinite line as compared with the volts at the same point in a line of indefinite length, we have by (1), Chap. II., $E_3 = Ee^{-\theta}$, where E_3 is the voltage at any point (say point B, Fig. 75) in an infinite line. If the line were disconnected at the same point the voltage would be $\frac{E}{\cosh \theta}$ by (13), Chap. III. The ratio of these voltages is

$$\frac{Ee^{-\theta}}{\frac{E}{\cosh \theta}} = e^{-\theta} \cosh \theta = \frac{(e^{\theta} + e^{-\theta})}{2} e^{-\theta} = \frac{1 + e^{-2\theta}}{2}$$

Now in an infinite line $e^{-\theta}$ may be neglected, and therefore the ratio becomes $\frac{1}{2}$, *i.e.* the voltage at point B in an infinite line is half that at the same point when the line is disconnected at that point, or, in other words, the voltage at a disconnected point is double that of the infinite line at the same point, which agrees with (1). When the line is earthed it is seen from para. 53 that the current is doubled as compared with the infinite line, and this agrees with (4). In both these cases the results are expressed in terms of the voltage and current at the point B in an infinite line, the value in the infinite line being taken as 1 for comparative purposes.

115. Partial Transmission and Reflection in an Infinite Transmission Line at the Junction of Unequal Impedances.—Fig. 75 illustrates this case. Equations (5) and (6) also apply, because the algebraic sum of the incident and reflected volts E_1 and E_2, or current I_1 and I_2, is always equal to the resultant or transmitted volts or current E_3 or I_3, and equations (5) and (6) are the mathematical statements of these facts. As regards the signs of voltage and current in this case, E_1 and I_1 are still always positive, and are

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fixed by the infinite line equation. If a fraction $m_1E_1 = E_3$ is transmitted, where m_1 is less than 1, then $(1-m_1)E_1 = \mp E_2$ is always reflected. We will, however, merely assume that $m_1E_1 = E_3$ and show how to obtain expressions for the coefficients of transmission and reflection by means of equations (5) and (6). The sign of E_2 may be + or —, and it may be determined as follows: The sign is + when Z_0 (Fig. 75) is greater than Z_a, for in that case the impedance at B is increased as compared with a uniform impedance Z_a throughout, and the voltage E_3 is also increased, *i.e.* it becomes greater than $Ee^{-\theta} = E_1$ owing to the fact that the impedance at B is now greater than the infinite line impedance Z_a. In that case E_2 must be positive. At the same time the current I_2 diminishes for the same reason, and is always opposite to E_2 in sign; the + sign meaning that the voltage or current at B, Fig. 75, is greater than would exist in the uniform infinite line Z_a, and the minus sign meaning that it would be less.

116. Coefficients of Transmission and Reflection.—From the preceding definitions, bearing in mind that equations (5) and (6) hold good for cases of partial reflection and transmission, equations will now be developed for the value of the coefficients of transmission and reflection expressed in terms of the impedances Z_a and Z_0. It must be borne in mind that E_2 is always opposite in sign to I_2.

To deduce the Coefficients of Transmission m_1 for Volts, and m for Current.

Case 1a.

$$E_3 = E_1 + E_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$I_3 = I_1 - I_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

At the same time we have, as may be deduced from Fig. 75 and the previous definitions—

$$\left.\begin{aligned} E_3 &= I_3Z_0 \\ E_2 &= I_2Z_a \\ E_1 &= I_1Z_a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let $E_3 = m_1E_1$ as stipulated in the preceding paragraph, and insert this value in (7),

then $$m_1E_1 = E_1 + E_2 \text{ or } E_2 = (m_1 - 1)E_1 \quad . \quad . \quad (10)$$

By the aid of (9) and (10) equation (8) may be written

$$\frac{E_3}{Z_0} = \frac{E_1}{Z_a} - \frac{E_2}{Z_a} = \frac{E_1}{Z_a} + \frac{(1-m_1)E_1}{Z_a} = \frac{E_1}{Z_a}(2-m_1)$$

that is $$\frac{E_3}{Z_0} = \frac{m_1E_1}{Z_0} = \frac{E_1}{Z_a}(2-m_1) \text{ or } m_1\left(\frac{1}{Z_0} + \frac{1}{Z_a}\right) = \frac{2}{Z_a}$$

whence $$m_1 = \frac{2Z_0}{Z_0 + Z_a} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is the coefficient of transmission for volts.

The coefficient of reflection is therefore

$$1 - m_1 = 1 - \frac{2Z_0}{Z_0 + Z_a} = \frac{Z_a - Z_0}{Z_a + Z_0} \quad . \quad . \quad (12)$$

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Case 1*b*.—The coefficient of transmission m for current is derived from (7) and (8) as follows :—

Let $I_3 = mI_1$

From (8) $mI_1 = I_1 - I_2$ or $I_2 = I_1(1-m)$

From (7) and (9) $I_3Z_0 = I_1Z_a + I_2Z_a$

and $\therefore mI_1Z_0 = I_1Z_a + I_1(1-m)Z_a$

Whence $mZ_0 = Z_a + (1-m)Z_a$ or $m(Z_0 + Z_a) = 2Z_a$

whence
$$m = \frac{2Z_a}{Z_0 + Z_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

This is the coefficient of transmission for the current when (7) and (8) hold good. The coefficient of reflection is therefore

$$1 - m = 1 - \frac{2Z_a}{Z_0 + Z_a} = \frac{Z_0 - Z_a}{Z_0 + Z_a} \quad . \quad . \quad (14)$$

Case 2 (*a*). To determine the coefficients of transmission and reflection

When $E_3 = E_1 - E_2$ . . . . . (15)

and $I_3 = I_1 + I_2$ . . . . . (16)

Let $E_3 = m_1E_1$ as before

Then by (15) $m_1E_1 = E_1 - E_2$ or $E_2 = E_1(1-m_1)$

From (16) and (9) $\frac{m_1E_1}{Z_0} = \frac{E_1}{Z_a} + \frac{E_1(1-m_1)}{Z_a}$ that is $\frac{m_1}{Z_0} = \frac{2-m_1}{Z_a}$

or $m_1\left(\frac{1}{Z_0} + \frac{1}{Z_a}\right) = \frac{2}{Z_a}$

whence
$$m_1 = \frac{2Z_0}{Z_0 + Z_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This is the same as (11), and therefore the coefficient of transmission for volts is the same whether E_2 has a plus or minus value. It follows that the coefficient of reflection for both cases is that given by (12).

Case 2 (*b*). The coefficient (m) for current is derived from (15) and (16) as follows :

Let $I_3 = mI_1$ . . . . . (18)

From (16) $mI_1 = I_1 + I_2$ or $I_2 = (m-1)I_1$

From (9) and (15) $I_3Z_0 = I_1Z_a - I_2Z_a$

whence $mI_1Z_0 = I_1Z_a - (m-1)I_1Z_a$ . . . . (19)

or $mZ_0 = Z_a + Z_a - mZ_a$

or $m(Z_0 + Za) = 2Za$

whence
$$m = \frac{2Z_a}{Z_0 + Z_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In this case also the coefficient of transmission has the same value as in (13), and therefore the coefficient is the same whether I_2 is

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positive or negative, and it follows that the coefficient of reflection for current is the same as (14), whether I_2 is positive or negative.

It will be seen, therefore, that there is only one coefficient of transmission and one of reflection for volts and for current respectively, at any given point.

If these equations be applied to the limiting impedances of zero and infinity the same results are obtained as previously deduced. For the resultant voltage with the line disconnected or short circuited we have by (11)

$$m_1 = \frac{2Z_0}{Z_a + Z_0} = \frac{2 \text{ Infinity}}{Z_a + \text{Infinity}} = 2$$

I.e. the resultant voltage is twice that which would be obtained on an infinite line at the same point. Similarly by (13) the current with the line disconnected is obtained. We have

$$m = \frac{2Z_a}{Z_a + Z_0} = \frac{2Z_a}{Z_a + \text{Infinity}} = 0$$

If the line is earthed at B we obtain the resultant volts by (11). We then have

$$m_1 = \frac{2Z_0}{Z_a + Z_0} = \frac{2 \times 0}{Z_a + 0} = 0$$

At the same time the resultant current may be obtained by (13). We then have

$$m = \frac{2Z_a}{Z_a + Z_0} = \frac{2Z_a}{Z_a + 0} = 2$$

These results agree with the previous deductions. A reference to equation (13) and para. 58, Chap. III., will show that an expression similar to the coefficient of transmission was developed from entirely different considerations.

The reflection and transmission formulæ deduced in the preceding equations will now be applied to two concrete cases to show the method of application.

It should be noted that, although formulæ (7) to (20) were formed from consideration of an infinite line, the formulæ are true for any length of line if all the reflections and transmissions are taken into account. The infinite line was utilized because it limits the reflections to one in each case. The following problem shows a method of applying the formulæ to a concrete case.

117. Formation of Transmission Equation by the Process of Reflection.—*To find the combined impedance of a transmission line of any length between two constant impedances.*—In formula (26), Chap. III., an expression is given for the receiving-end impedance of a transmission line of any length with similar apparatus at both ends. The following problem is identical with that one in its main features, as will now be shown.

Fig. 76 represents the case of a short line in the middle of two indefinitely long lines, and we will now develop a formula which permits of the attenuation in the circuit being computed. A further

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aspect of the problem is to find a formula for the transmission loss between A and D as compared with the transmission loss which would exist if $Za = Z_0$. Za and Z_0 are characteristic impedances. Let Z_0 be variable and Za constant.

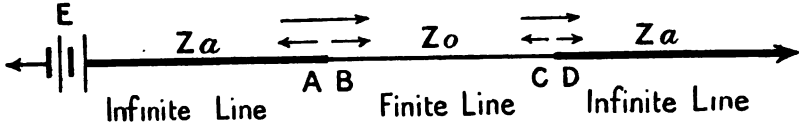

FIG. 76.

Transmitted Current.—If the current incident at a point A in the infinite line Za is I_2,

The current at B becomes $\frac{2Za}{Z_0 + Za} \times I_2$ (by (20)).

The current arriving at C from B is $I_2 \frac{2Za}{Z_0 + Za} e^{-\theta}$, call it I_3.

Then the current at C being I_3 the current at D is $\frac{2Z_0}{Z_0 + Za} \times I_3$.

Note that Za and Z_0 are in turn used in the numerator of the transmission coefficient; the impedance of the section preceding the change of impedance to which the formula refers always appears in the numerator of m.

Reflected Current.—The current on arriving at C splits up into two parts, the transmitted current (already dealt with) and the reflected portion $I_3 \frac{Za - Z_0}{Za + Z_0}$, which is reflected back to B, where it has the value $I_3 \frac{Za - Z_0}{Za + Z_0} e^{-\theta}$, the portion $\frac{Za - Z_0}{Za + Z_0}$ of this current is again reflected back to C, and it there becomes $I_3 \left(\frac{Za - Z_0}{Za + Z_0}\right)^2 e^{-2\theta}$.

This process of reflection is indefinitely repeated.

Summation of Reflected and Incident Currents.—If we designate $\left(\frac{Za - Z_0}{Za + Z_0}\right)^2 e^{-2\theta}$ by x, then y, the sum of the current at C due to the incident current, and repeated reflections, will be

$$\begin{aligned} y &= I_3 + I_3 x + I_3 x^2 + I_3 x^3 + \ldots \\ &= I_3 (1 + x + x^2 + x^3 + \ldots) \\ &= I_2 \frac{2Za}{Za + Z_0} e^{-\theta} (1 + x + x^2 + x^3 + \ldots) \end{aligned} \qquad (21)$$

The quantity in the brackets is a geometrical progression, and x is less than 1. By (14), Chap. I., the sum of an infinite number of terms is $\frac{a}{1 - x}$. If $a = 1$ the total current at C may therefore be written

$$I_3 = I_2 \frac{2Za}{Za + Z_0} e^{-\theta} \frac{1}{1 - x} \quad . \quad . \quad . \quad . \quad (22)$$

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At the point D this becomes

$$I_4 = I_3\frac{2Z_0}{Za+Z_0} = I_2\frac{2Z_0}{Za+Z_0}\cdot\frac{2Za}{Za+Z_0}e^{-\theta}\frac{1}{1-x} \text{ by (22)}$$

$$= I_2\frac{4ZaZ_0}{(Za+Z_0)^2}e^{-\theta}\frac{1}{1-x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now it can be shown without difficulty that this formula gives the same result as (23), Chap. III. This latter formula was, of course, developed from different considerations. To prove that the two formulæ lead to the same result it will be convenient to deal with the impedance corresponding to (23) as here developed. If the voltage at A, Fig. 76, is E_1, and the current at D is I_4 as in (23), we have $\frac{E_1}{I_4} = Z_4$, and from (23) above we obtain

$$Z_4 = \frac{E_1}{I_2}\frac{(Za+Z_0)^2}{4ZaZ_0}e^{\theta}(1-x) \quad . \quad . \quad . \quad (24)$$

Substitute the value of x and we obtain

$$Z_4 = \frac{E_1}{I_2}\frac{(Za+Z_0)^2}{4ZaZ_0}e^{\theta}\left\{1-\left(\frac{Za-Z_0}{Za+Z_0}\right)^2e^{-2\theta}\right\} \quad . \quad (25)$$

$$= \frac{E_1}{I_2}e^{\theta}\left\{\frac{(Za+Z_0)^2}{4ZaZ_0} - \frac{(Za+Z_0)^2}{4ZaZ_0}\left(\frac{Za-Z_0}{Za+Z_0}\right)^2e^{-2\theta}\right\}$$

$$= \frac{E_1}{I_2}e^{\theta}\left\{\frac{(Za+Z_0)^2}{4ZaZ_0} - \frac{(Za-Z_0)^2}{4ZaZ_0}e^{-2\theta}\right\}$$

$$= \frac{E_1}{I_2}e^{\theta}\left\{\frac{Za^2+Z_0^2}{4ZaZ_0} + \frac{2ZaZ_0}{4ZaZ_0} - e^{-2\theta}\left(\frac{Za^2+Z_0^2}{4ZaZ_0} - \frac{2ZaZ_0}{4ZaZ_0}\right)\right\} \quad (26)$$

This impedance is developed in para. 62, and is given by (26), Chap. III., and the latter is easily converted to (26) above: thus from (26), Chap. III.,

$$Z_2 = \frac{Za^2+Z_0^2}{Z_0}\sinh\theta + 2Za\cosh\theta$$

$$= \frac{Za^2+Z_0^2}{Z_0}\frac{e^{\theta}}{2} - \frac{Za^2+Z_0^2}{Z_0}\frac{e^{-\theta}}{2} + 2Za\frac{e^{\theta}}{2} + 2Za\frac{e^{-\theta}}{2}$$

$$= \left(\frac{Za^2+Z_0^2}{Z_0} + 2Za\right)\frac{e^{\theta}}{2} - \left(\frac{Za^2+Z_0^2}{Z_0} - 2Za\right)\frac{e^{-\theta}}{2}$$

$$= 2Zae^{\theta}\left\{\frac{Za^2+Z_0^2}{4ZoZa} + \frac{2ZaZ_0}{4ZaZ_0} - e^{-2\theta}\left(\frac{Za^2+Z_0^2}{4ZaZ_0} - \frac{2ZaZ_0}{4ZaZ_0}\right)\right\} \quad (27)$$

Now (26) and (27) only differ in the fact that $\frac{E_1}{I_2}$ in (26) is replaced by $2Za$ in (27). This difference is due to the end conditions of the circuits under consideration. A reference to the relative circuit diagrams will show the difference (see Figs. 18 and 76). In the one case a line of any length is terminated at each end by apparatus having the same impedance, whilst in the other case a line of any length is terminated at each end by similar infinite

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lines. In each instance a line of any length is thus inserted between two constant impedances, and that portion of (26) and (27) in the outer brackets shows how the received current at D in Fig. 76, and in the receiving apparatus in Fig. 18, is affected by the fact that the impedance of the inserted line differs from that of the two impedances between which it is inserted. It is evident by inspection of (26) and (27) that if $Z_0 = Za$ the whole quantity in the outer brackets reduces to 1, this assumes that Za and Z_0 have the same angle; we then have the condition for no reflection and the equation reduces to that of the infinite line. This equation may be written as follows :—

$$Z_2 = 2Zae^{\theta}\left[\frac{1}{4}\left(\frac{Za}{Z_0}+\frac{Z_0}{Za}\right)+\frac{1}{2}+e^{-2\theta}\left\{\frac{1}{2}-\frac{1}{4}\left(\frac{Za}{Z_0}+\frac{Z_0}{Za}\right)\right\}\right] \quad (28)$$

The effect of the angle of the vector impedances Za and Z_0 in this case is dealt with in para. 121.

An equation similar to (28) was given by Monsieur Devaux-Charbonnel in "Les Cables Téléphoniques Soumarins," in *La Lumière Electrique*, 15th June, 1912. See also Prof. Breisig's "Theoretische Télégraphie," p. 302, and "Note sur le Calcul des Reflexions dans les Lignes Composées," by M. H. Pleijel in the "*Comptes Rendus* de la Conference Internationale des Techniciens des Administrations des Télégraphes et des Téléphones de l'Europe:" Paris: Imprimerie Nationale, 1910. The subject is also dealt with in Dr. Kennelly's "Application of Hyperbolic Functions to Engineering Problems," chap. vi., and Prof. Fleming's "The Propagation of Electric Currents in Telegraph and Telephone Conductors," p. 63.

It is evident from the consideration of (26) and (27) that the same result is obtained whether the formula be developed by the method of Chapter III., and Appendix III., or built up by the reflection process.

118. Reflection at the Ends of a Finite Transmission Line open at one End and closed at the other. Expression for Volts at the

FIG. 77.

Receiving End.—The preceding example shows a method of dealing with reflection at the boundary of unequal impedances, the following example shows the case of a finite line as depicted in Fig. 77. The earthed generator E has no resistance, and the disconnected line AB is of finite length. The example is given by Dr. Kennelly in the chapter quoted. The problem is to find the voltage at the open end of the line. The first voltage wave by hypothesis acts as if it were in an infinite line. The voltage E, after transmission to B, has the value $Ee^{-\theta}$, by (1), Chap. II. It is reflected there and has then the value $2Ee^{-\theta}$ by (1). The reflected

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wave $Ee^{-\theta}$ then travels back to A, where the incident voltage is $Ee^{-2\theta}$, but the reflected wave is $-Ee^{-2\theta}$ by (3). On arrival again at B the voltage increment is $-Ee^{-3\theta}$, and, as before, this quantity is doubled at that point and becomes $-2Ee^{-3\theta}$. The reflected part $-Ee^{-3\theta}$ is transmitted to A, where it has the value $-Ee^{-4\theta}$ before further reflection. It travels back after reflection as $Ee^{-4\theta}$, and on arrival at B after reflection has the value $2Ee^{-5\theta}$. This process is theoretically continued to infinity, the total volts E_1 at B being the sum of an infinite series which has the value

$$E_1 = 2E(e^{-\theta} - e^{-3\theta} + e^{-5\theta} - e^{-7\theta} + \ldots)$$
$$= 2Ee^{-\theta}(1 - e^{-2\theta} + e^{-4\theta} - e^{-6\theta} + \ldots) \quad . \quad (29)$$

The series in the brackets is a geometrical progression, and by (14), Chap. I., the sum to infinity is $\frac{a}{1-r}$, where $a = 1$ and r in this case has the value $-e^{-2\theta}$ and therefore $\frac{1}{1-r} = \frac{1}{1+e^{-2\theta}}$.

Therefore (29) has the value

$$E_1 = 2E\frac{e^{-\theta}}{1+e^{-2\theta}} = \frac{2E}{e^{\theta}+e^{-\theta}}$$

or

$$E_1 = \frac{E}{\cosh \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Which agrees with (13), Chap. III.

By the application of formulæ (1) to (20) numerous transmission equations may be developed, but the more important ones for our purpose have been already developed by other means, and it is not necessary to repeat them. It is seen from the preceding investigation that some light is thrown on transmission losses at the junction of unequal impedances by taking reflection into account.

Now it is possible to investigate such losses from quite a different point of view, *i.e.* by the application of energy equations, to find the conditions for maximum power at the junction of impedances in transmission lines, and as the method is relatively simple, and the subject of great importance, it will now be dealt with.

119. Relations between the Power in Terminal Apparatus at the Ends of a Uniform Transmission Line and the Relative Impedance of the Line and Apparatus. Introductory Remarks.—In order to understand the transmission phenomena which occur in a telephone circuit with speaking apparatus at either or both ends of the line, it is necessary to have a clear idea of the effects of changes in the vector impedance of the apparatus and line, on the power in the circuit, and the conditions which govern the maximum power in the apparatus at either end of the line. This will now be considered as a power problem, without taking reflection into account. The following cases will be dealt with :—

(*a*) A uniform transmission line with apparatus at the receiving end.

(*b*) A uniform transmission line with apparatus at the sending end.

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(*c*) A uniform transmission line with similar apparatus at both ends of the line.

(*d*) A uniform transmission line with apparatus of different impedance at the two ends.

It may be remarked at the outset that the mathematical expression which shows the relation between the vector impedances of the line and apparatus and determines the power in the apparatus, is the same in its main essentials in the first two cases, and the elucidation of the fourth case (*d*) results in the conclusion that the best result is obtained with similar apparatus at both ends of the line and comes in the same general category as the other cases. Case (*a*) will therefore be examined as fully as appears to be necessary in the first instance, and the other cases will be examined sufficiently to make it clear that the same general conclusion applies to them. *In all cases the impedances are vectors, but for convenience the angles* (ϕ, ϕ_1, etc.) *are only introduced when necessary.*

Case (*c*) refers to current only, but the mathematical expression deduced is covered by case (*a*), and it is therefore included.

120. Power in the Receiving Apparatus and Adjacent Line in a Transmission Circuit of any Length closed through the Apparatus at the Receiving End.—*Case* (*a*).—Fig. 16, Chap. III., illustrates this case. The line without the apparatus has a resistance $Z_1 = Z_0 \tanh \theta$ by (6), Chap. III., and it is now required to find what impedance Za must be given to the receiving apparatus, taking into account that Za and Z_0 are vectors, in order that it may absorb the maximum amount of power with a fixed voltage E_1 at the sending end, and if nothing changes but Za. By (17), Chap. III., we have

$$I_2 = \frac{E_1}{Z_0 \sinh \theta + Za \cosh \theta}$$

The power in the apparatus Za will be

$$P = I_2^2 Za = \frac{E_1^2 Za}{(Z_0 \sinh \theta + Za \cosh \theta)^2} \quad . \quad . \quad (31)$$

It is convenient to operate on the reciprocal of P, and we thus obtain

$$\frac{1}{P} = \frac{Z_0(Z_0 \sinh \theta + Za \cosh \theta)^2}{E_1^2 Za Z_0} \quad . \quad . \quad . \quad (32)$$

$$= \frac{Z_0(Z_0^2 \sinh^2 \theta + Za^2 \cosh^2 \theta + 2Z_0 Za \cosh \theta \sinh \theta)}{E_1^2 Za Z_0}$$

$$= \frac{Z_0}{E_1^2}\left(\frac{Z_0^2 \sinh^2 \theta + Za^2 \cosh^2 \theta}{Za Z_0} + 2 \cosh \theta \sinh \theta\right)$$

$$= \frac{Z_0}{E_1^2}\left\{\left(\frac{Z_0 \sinh^2 \theta}{Za} + \frac{Za \cosh^2 \theta}{Z_0}\right) + 2 \sinh \theta \cosh \theta\right\} \quad (33)$$

Let the impedance Z_0 be written $Z_0 \underline{/\phi}$ and Za become $Za \underline{/\phi_1}$ where both Z_0 and $\underline{/\phi}$ and Za and $\underline{/\phi_1}$ may have any value.

It will be convenient at this stage to write a for $\sinh^2 \theta$ and b for $\cosh^2 \theta$, formula (33) then becomes

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$$\frac{I}{P} = \frac{Z_0/\phi}{E_1^2}\left\{\left(a\frac{Z_0/\phi}{Za/\phi_1} + b\frac{Za/\phi_1}{Z_0/\phi}\right) + 2\sqrt{ab}\right\} \quad . \quad (34)$$

By (18), Chap. I., this may be written

$$\frac{I}{P} = \frac{Z_0/\phi}{E_1^2}\left\{\left(a\frac{Z_0 e^{j\phi}}{Za e^{j\phi_1}} + b\frac{Za e^{j\phi_1}}{Z_0 e^{j\phi}}\right) + 2\sqrt{ab}\right\}$$

and by (24), Chap. I., this is

$$\frac{I}{P} = \frac{Z_0/\phi}{E_1^2}\left[\left\{a\frac{Z_0}{Za}e^{j(\phi-\phi_1)} + b\frac{Za}{Z_0}e^{j(\phi_1-\phi)}\right\} + 2\sqrt{ab}\right] \quad (35)$$

Now $j(\phi_1 - \phi) = -j(\phi - \phi_1)$ and therefore (35) may be written

$$\frac{I}{P} = \frac{Z_0/\phi}{E_1^2}\left[\left\{a\frac{Z_0}{Za}e^{j(\phi-\phi_1)} + b\frac{Za}{Z_0}e^{-j(\phi-\phi_1)}\right\} + 2\sqrt{ab}\right] \quad (36)$$

121. Effects of Impedance Changes on the Received Power in a Transmission Line.—These effects may be elucidated as follows. From (7), Chap. I., (36) may be written—

$$\frac{I}{P} = \frac{Z_0/\phi}{E_1^2}\left[a\frac{Z_0}{Za}\{\cos(\phi-\phi_1) + j\sin(\phi-\phi_1)\}\right.$$
$$\left. + b\frac{Za}{Z_0}\{\cos(\phi-\phi_1) - j\sin(\phi-\phi_1)\} + 2\sinh\theta\cosh\theta\right]$$
$$= \frac{Z_0/\phi}{E_1^2}\left\{a\frac{Z_0}{Za}\cos(\phi-\phi_1) + a\frac{Z_0}{Za}j\sin(\phi-\phi_1)\right.$$
$$\left. + b\frac{Za}{Z_0}\cos(\phi-\phi_1) - b\frac{Za}{Z_0}j\sin(\phi-\phi_1) + 2\sinh\theta\cosh\theta\right\}$$
$$= \frac{Z_0/\phi}{E_1^2}\left\{\left(a\frac{Z_0}{Za} + b\frac{Za}{Z_0}\right)\cos(\phi-\phi_1)\right.$$
$$\left. + \left(a\frac{Z_0}{Za} - b\frac{Za}{Z_0}\right)j\sin(\phi-\phi_1) + 2\sinh\theta\cosh\theta\right\}$$

To separate the Angles from the Modulus.—If only Za varies construct a diagram of the vector quantities containing the variable Za, thus

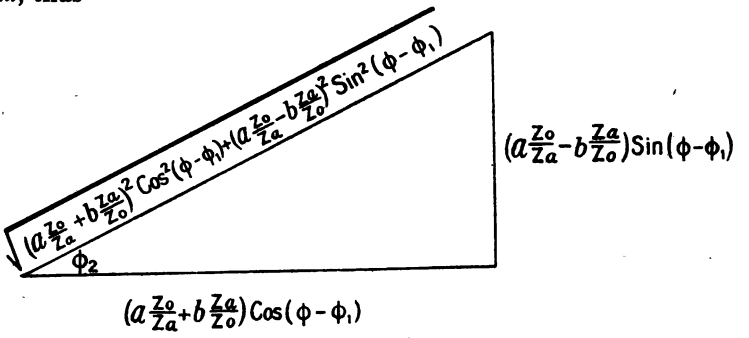

FIG. 78.

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(36) may now be written

$$\frac{I}{P}=\frac{Z_0/\phi}{E_1^2}\left[\left\{\sqrt{\left(a\frac{Z_0}{Za}+b\frac{Za}{Z_0}\right)^2\cos^2(\phi-\phi_1)+\left(a\frac{Z_0}{Za}-b\frac{Za}{Z_0}\right)\sin^2(\phi-\phi_1)}\right\}\right.$$
$$\left.\underline{|\tan^{-1}\phi_2+2\sqrt{ab}}\right] \quad . \quad (37)$$

Squaring the quantities in the brackets under the root sign we have

$$=\frac{Z_0/\phi}{E_1^2}\left[\left\{\sqrt{\left\{\left(a^2\frac{Z_0^2}{Za^2}+b^2\frac{Za^2}{Z_0^2}+2ab\right)\cos^2(\phi-\phi_1)+\right.\right.\right.$$
$$\left.\left.\left.+\left(a^2\frac{Z_0^2}{Za^2}+b^2\frac{Za^2}{Z_0^2}-2ab\right)\sin^2(\phi-\phi_1)\right\}}\right\}\underline{|\tan^{-1}\phi_2+2\sqrt{ab}}\right]$$
$$=\frac{Z_0/\phi}{E_1^2}\left[\left\{\sqrt{\left[a^2\frac{Z_0^2}{Za^2}\left\{\cos^2(\phi-\phi_1)+\sin^2(\phi-\phi_1)\right\}+\right.\right.\right.$$
$$\left.\left.\left.+2ab\left\{\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)\right\}+b^2\frac{Za^2}{Z_0^2}\left\{\cos^2(\phi-\phi_1)+\sin^2(\phi-\phi_1)\right\}\right]}\right.\right.$$
$$\left.\left.\underline{|\tan^{-1}\phi_2+2\sqrt{ab}}\right\}\right] \quad . \quad (38)$$

Now $\cos^2(\phi-\phi_1)+\sin^2(\phi-\phi_1)=1$, and therefore (38) may be written

$$=\frac{Z_0/\phi}{E_1^2}\left\{\sqrt{\left[\left(a^2\frac{Z_0^2}{Za^2}+b^2\frac{Za^2}{Z_0^2}\right)+2ab\left\{\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)\right\}\right]}\right.$$
$$\left.\underline{|\tan^{-1}\phi_2+2\sqrt{ab}}\right\} \quad . \quad (39)$$

This gives the same result as (34), but it gives the moduli and angles separately and the effects will now be investigated.

Minimum Value of the Modulus.—The only variable Za in (39) is in the expression $\left(a^2\frac{Z_0^2}{Za^2}+b^2\frac{Za^2}{Z_0^2}\right)$

If the constant quantity a^2Zo^2 is called a_1, and the constant quantity $\frac{b^2}{Z_0^2}$ is called b_1, the expression becomes $\left(\frac{a_1}{Za^2}+b_1Za^2\right)$, which is similar to (89), Chap. I., and therefore by (90), Chap. I., it is a minimum when

$$\frac{a^2Z_0^2}{Za^2}=\frac{b^2Za^2}{Z_0^2} \quad \text{or} \quad \sqrt{\frac{a}{b}}=\frac{Za}{Z}$$

But by definition

$$\sqrt{\frac{a}{b}}=\sqrt{\frac{\sinh^2\theta}{\cosh^2\theta}}=\tanh\theta=\frac{Z}{Z_0} \quad . \quad . \quad (40)$$

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Insert this value of $\frac{Za}{Z_0}$ in (39) and we then obtain

$$a^2\frac{Z_0^2}{Za^2}+b^2\frac{Za^2}{Z_0^2}=\sinh^4\theta\frac{\cosh^2\theta}{\sinh^2\theta}+\cosh^4\theta\frac{\sinh^2\theta}{\cosh^2\theta}$$
$$=2\sinh^2\theta\cosh^2\theta=\frac{\sinh^2 2\theta}{2} \quad . \quad (41)$$

It should be noticed that since $\frac{Za}{Z_0}=\tanh\theta$ the angle $\phi_2=0$ since by substitution of $\tanh\theta$ in the vertical of Fig. 78 that component is 0.

By substitution of (41) in (39) we obtain

$$\frac{I}{P}=\frac{Z_0/\phi}{E_1^2}\left\{\sqrt{\left[\frac{\sinh^2 2\theta}{2}+\frac{\sinh^2 2\theta}{2}\left\{\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)\right\}\right]}+\sinh 2\theta\right\} \quad . \quad (42)$$

If $(\phi-\phi_1)=0$, this becomes

$$\frac{Z_0/\phi}{E_1^2}\left(\sqrt{\frac{\sinh^2 2\theta}{2}+\frac{\sinh^2 2\theta}{2}}+\sinh 2\theta\right)=\frac{Z_0/\phi}{E_1^2}2\sinh 2\theta$$

The effect of the angle may now be seen as follows.

If the angle $(\phi-\phi_1)$ is 90°, then $\cos(\phi-\phi_1)=0$, and $\sinh(\phi-\phi_1)=1$ and $\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)=-1$; therefore (42) becomes

$$\frac{I}{P}=\frac{Z_0/\phi}{E_1^2}\left\{\sqrt{\left(\frac{\sinh^2 2\phi}{2}-\frac{\sinh^2 2\phi}{2}\right)}+\sinh 2\theta\right\}$$

or

$$\frac{I}{P}=\frac{Z_0/\phi}{E_1^2}\sinh 2\theta \quad . \quad . \quad . \quad . \quad (43)$$

The effect of the angle in this case is to double the power P as compared with (42) *et seq.* A practical case occurs where the angle $(\phi-\phi_1)=90°$ approximately, when the standard apparatus (see Fig. 135) is connected to the standard cable.

Now (42) and (43) are for lines of any length, and it is evident that the value of (43) for the case of a long line may be written

$$\frac{I}{P}=\frac{Z_0/\phi}{2E_1^2}e^{2\theta} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

General Formulæ for $\frac{I}{P}$ in the Case of a Long Line.—If the values of a and b are inserted in (39) we obtain

$$\frac{I}{P}=\frac{Z_0/\phi}{E_1^2}\left\{\sqrt{\left[\left(\sinh^4\theta\frac{Z_0^2}{Za^2}+\cosh^4\theta\frac{Za^2}{Z_0^2}\right)+2\sinh^2\theta\cosh^2\theta\left\{\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)\right\}\right]}+\tan^{-1}\phi_2+2\sinh\theta\cosh\theta\right\} \quad . \quad . \quad . \quad (45)$$

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If the line is very long this becomes

$$\frac{I}{P}=\frac{Z_0 \angle \phi}{E_1^2}\left\{\sqrt{\left[\left\{\frac{(e^\theta)^4 Z_0^2}{16\, Za^2}+\frac{(e^\theta)^4 Za^2}{16\, Z_0^2}\right\}+\right.\right.}$$

$$\left.\sqrt{+2\,\frac{(e^\theta)^2}{4}\times\frac{(e^\theta)^2}{4}\left\{\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)\right\}}\right.$$

$$\left.\left.\underline{|\tan^{-1}\phi_2}+2\,\frac{e^\theta}{2}\times\frac{e^\theta}{2}\right\}\right]$$

$$=\frac{Z_0 \angle \phi\, e^{2\theta}}{E_1^2}\left[\frac{1}{4}\left\{\sqrt{\left[\frac{Z_0^2}{Za^2}+\frac{Za^2}{Z_0^2}+2\left\{\cos^2(\phi-\phi_1)-\sin^2(\phi-\phi_1)\right\}\right]}\right.\right.$$

$$\left.\left.\underline{|\tan^{-1}\phi_2}+2\right\}\right] \quad . \quad (46)$$

If $Z_0 = Za$ and $(\phi - \phi_1) = 0$ this becomes

$$\frac{Z_0 \angle \phi\, e^{2\theta}}{E_1^2}\left[\frac{1}{4}\left\{\sqrt{(1+1+2)}+2\right\}\right]=\frac{Z_0 \angle \phi e^{2\theta}}{E_1^2}$$

If $(\phi - \phi_1) = 45°$ and the line is long, a useful formula is obtained for the case where the standard apparatus is used on loaded lines, for the angle of the apparatus is 45°, and the loaded line has a very small angle.

If $(\phi - \phi_1) = 45°$, $\cos^2(\phi - \phi_1) - \sin^2(\phi - \phi_1) = \cos^2 45° - \sin^2 45° = \frac{1}{2} - \frac{1}{2} = 0$. Formula (46) then becomes

$$\frac{I}{P}=\frac{Z_0 \angle \phi\, e^{2\theta}}{E_1^2}\left[\frac{1}{4}\left\{\sqrt{\left(\frac{Z_0^2}{Za^2}+\frac{Za^2}{Z_0^2}\right)}\underline{|\tan^{-1}\phi_2}+2\right\}\right] \quad . \quad (47)$$

If in (46) $\phi - \phi_1 = 90$ and $Za = Z_0$ we have

$$\frac{I}{P}=\frac{Z_0 \angle \phi\, e^{2\theta}}{E_1^2}\left[\frac{1}{4}\left\{\sqrt{(1+1-2)}+2\right\}\right]=\frac{Z_0 \angle \phi\, e^{2\theta}}{2E_1^2} \quad . \quad (48)$$

This last result, and other cases where $Z_0 = Za$, may be illustrated vectorially, for the quantity under the root sign in (46) is a special case of the quantity under the root sign in (39), which is the equivalent of the quantity in the inner brackets in (36). If this latter expression be modified for the case of a long line, we obtain

$$y=\frac{e^{2\theta}}{4}\left(\frac{Z_0}{Za}e^{j(\phi-\phi_1)}+\frac{Za}{Z_0}e^{-j(\phi-\phi_1)}\right) \quad . \quad . \quad (49)$$

If $Za = Z_0$ the quantity in the brackets may be written

$$y = \cos(\phi - \phi_1) + j\sin(\phi - \phi_1) + \cos(\phi - \phi_1) - j\sin(\phi - \phi_1)$$
$$= 2\cos(\phi - \phi_1)$$

If for convenience we write ϕ_3 for $(\phi - \phi_1)$ we have

$$y = (\cos\phi_3 + j\sin\phi_3 + \cos\phi_3 - j\sin\phi_3) = 2\cos\phi_3 \quad (50)$$

Fig. 79 illustrates this case.

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Two similar vectors $e^{j\phi_3}$ and $e^{-j\phi_3}$ are here represented. If we imagine the two vectors to revolve in opposite directions with equal speed and the angle ϕ_3 to steadily increase from 0° to 180,° the two vectors are equal and the sum of the angles is always $2 \cos \phi_3$. If the angle $= 0°$ the vectors are together along the horizontal line from the centre to the right. If $\phi_3 = 90°$ the vectors are both in the vertical position, and $2 \cos \phi_3 = 0$. This means that when $Za = Z_0$, and the angle is 90°, the quantity under the square root sign in equation (46) $= 0$, which corresponds with the result obtained in (48), as denoted by the quantity under the root sign. When the angle is 0 then $2 \cos \phi_3 = 2$, and if the value of the square root factor in (46) be taken when $(\phi - \phi_1) = 0$, this will be found

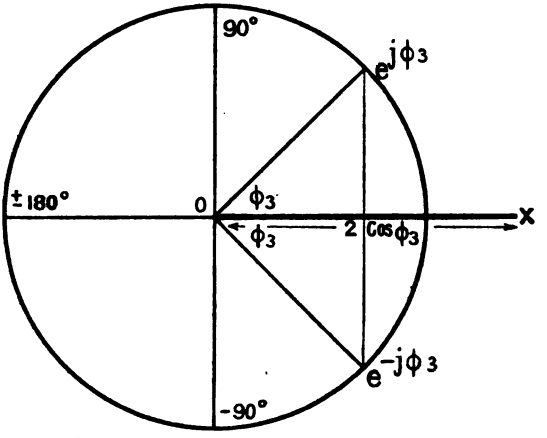

FIG. 79.

to give the same result, as shown by the example following (46). Similarly if the angle ϕ_3 is 45°, then $2 \cos \phi_3 = 2 \times .707 = 1.414$. This is the case given in (47), and if $Z_0 = Za$ the square root of the quantity under the root sign is $\sqrt{2} = 1.414$ in agreement with Fig. 79.

If the vectors are rotated into the second and third quadrants $2 \cos \phi_3$ becomes negative, and if they are rotated to the angle of 180°, $2 \cos \phi_3 = -2$. The physical limitations of the problems which are likely to require solution here do not appear to involve an angle of $(\phi - \phi_1) = \phi_3$ greater than $/135°$, *i.e.* a limiting angle of apparatus impedance of $/90°$ and a line of characteristic impedance with an angle of $\backslash 45°$.

122. Power in the Sending Apparatus and Adjacent Line in a Transmission Circuit of any Length closed at the Receiving End without Apparatus at that Point.—*Case* (*b*).—Fig. (16), Chap. III., illustrates the case, but in this instance the battery is applied at the other end of the line. The resistance of the apparatus is *Za*, and that of the line $Z_0 \tanh \theta$ by (6), Chap. III. The sent current is therefore

$$I_1 = \frac{E_1}{Za + Z_0 \tanh \theta} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

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The power in the sending apparatus Za is

$$P = I_1^2 Za = \frac{E_1^2 Za}{(Za + Z_0 \tanh \theta)^2} \quad . \quad . \quad . \quad (52)$$

$$\therefore \frac{1}{P} = \frac{Z_0(Za + Z_0 \tanh \theta)^2}{E_1^2 Za Z_0} \quad . \quad . \quad . \quad (53)$$

$$= \frac{Z_0(Za^2 + Z_0^2 \tanh^2 \theta + 2ZaZ_0 \tanh \theta)}{E_1^2 Za Z_0}$$

$$= \frac{Z_0}{E_1^2}\left\{\left(\frac{Za}{Z_0} + \frac{Z_0 \tanh^2 \theta}{Za}\right) + 2 \tanh \theta\right\}$$

$$= \frac{Z_0}{E_1^2 \cosh^2 \theta}\left\{\left(\frac{Za}{Z_0}\cosh^2 \theta + \frac{Z_0 \sinh^2 \theta \cosh^2 \theta}{Za \cosh^2 \theta}\right) + \frac{2 \sinh \theta \cosh^2 \theta}{\cosh \theta}\right\}$$

$$= \frac{Z_0}{E_1^2 \cosh^2 \theta}\left\{\left(\frac{Z_0 \sinh^2 \theta}{Za} + \frac{Za \cosh^2 \theta}{Z_0}\right) + 2 \sinh \theta \cosh \theta\right\} \quad (54)$$

Now the quantity in the outer brackets in (54) is the same as that in the outer brackets in (33). In each of these formulæ the quantities outside the brackets are constants for the purposes of this investigation, and as the determination of the maximum power depends upon the quantities within the *inner* brackets, the two formulæ lead to the same general result.

123. Terminal Loss in Transmission Lines.—*Case* (*c*).—In this case the problem is to find the effect on the received current of a difference in the impedances of the apparatus and the line, in a long circuit with similar apparatus at both ends. It will be shown that the required formula in this case can be reduced to an expression covered by the investigation in case (*a*).

In order to develop an equation to express the so-called terminal loss formula it will be convenient to adapt (23), Chap. III., to the required conditions, when the meaning will be easily shown. We have here the case of a circuit with similar apparatus at both ends. In that case

$$I'_2 = \frac{E_1}{\frac{Za^2 + Z_0^2}{Z_0}\sinh \theta + 2Za \cosh \theta} = \frac{E_1 Z_0}{(Za^2 + Z_0^2) \sinh \theta + 2ZaZ_0 \cosh \theta}$$

If the line is indefinitely long this may be written

$$I'_2 = \frac{E_1 Z_0}{\frac{e^\theta}{2}(Za^2 + Z_0^2 + 2ZaZ_0)} = \frac{2E_1 Z_0}{e^\theta (Za + Z_0)^2} \quad . \quad (55)$$

This is the general expression for the current at the receiving end of a sufficiently long line.

If $Za = Z_0$ (55) becomes

$$I_2 = \frac{E_1 e^{-\theta}}{2Z_0} = \frac{E_1 e^{-\theta}}{2Za} \quad . \quad . \quad . \quad . \quad (56)$$

If (55) is multiplied and divided by $2Za$ it becomes

$$I'_2 = \frac{E_1 e^{-\theta}}{2Za} \cdot \frac{4ZaZ_0}{(Za + Z_0)^2} \quad . \quad . \quad . \quad . \quad (57)$$

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This equation has evidently the same value as (55), and since (55) and (57) give the same result we have

$$\frac{I'_2}{I_2} = \frac{\frac{2E_1Z_0e^{-\theta}}{(Za+Z_0)^2}}{\frac{E_1e^{-\theta}}{2Za}} = \frac{4ZaZ_0}{(Za+Z_0)^2} \quad . \quad . \quad . \quad (58)$$

and therefore

$$\frac{\frac{I}{I'_2}}{\frac{I}{I_2}} = \frac{Z'_2}{Z_2} = \frac{(Za+Z_0)^2}{4ZaZ_0} \quad . \quad . \quad . \quad . \quad (59)$$

This may be written

$$\frac{Za^2+Z_0^2}{4ZaZ_0} + \frac{2ZaZ_0}{4ZaZ_0} = \left(\frac{Za^2+Z_0^2}{4ZaZ_0} + \frac{1}{2}\right) = \frac{1}{4}\left(\frac{Za}{Z_0} + \frac{Z_0}{Za} + 2\right) \quad (60)$$

If the angle be expressed, the formula for terminal loss is the quantity in the outer brackets in (46). If we equate this quantity thus

$$e^{\theta_t} = \left[\frac{1}{4}\left(\sqrt{\left\{\frac{Z_0^2}{Za^2} + \frac{Za^2}{Z_0^2} + 2\{\cos^2(\phi-\phi_1) - \sin^2(\phi-\phi_1)\}\right\}} \right.\right.$$
$$\left.\left. /\tan^{-1}\phi_2 + 2\right)\right] \quad (61)$$

then θ_t may be treated as an attenuation, and (61) is the same as (59) if Za and Z_0 are vectors in the latter equation.

124. Power in the Receiving Apparatus and Adjacent Line at the End of a Transmission Line of any Length, the Line having Apparatus joined to it at both Ends.—*Case (d).*—Fig. 19 illustrates this case. The resistance of the receiving apparatus is Zb, and by (30), Chap. III., the received current in Zb is

$$I_2 = \frac{E_1}{\left(\frac{Z_0^2+ZaZb}{Z_0}\right)\sinh\theta + (Za+Zb)\cosh\theta}$$

Now Za and Zb are vectors, the power in the receiving apparatus $Zb = P = I_2^2Zb$, therefore writing $\frac{I}{P}$ as before, we have

$$\frac{I}{P} = \frac{I}{E_1^2Zb}\left\{\left(\frac{Z_0^2+ZaZb}{Z_0}\right)\sinh\theta + (Za+Zb)\cosh\theta\right\}^2 \quad (62)$$

To find the minimum value bring this to the form of (89), Chap. I., as follows :—

$$\frac{I}{P} = \frac{I}{E_1^2}\left\{\frac{(Z_0^2+ZaZb)}{Z_0\sqrt{Zb}}\sinh\theta + \frac{(Za+Zb)}{\sqrt{Zb}}\cosh\theta\right\}^2$$
$$= \frac{I}{E_1^2}\left\{\frac{Z_0^2\sinh\theta + ZaZb\sinh\theta + Z_0Za\cosh\theta + Z_0Zb\cosh\theta}{Z_0\sqrt{Zb}}\right\}^2$$
$$= \frac{I}{E_1^2}\left\{\frac{Z_0(Z_0\sinh\theta + Za\cosh\theta) + Zb(Za\sinh\theta + Z_0\cosh\theta)}{Z_0\sqrt{Zb}}\right\}^2$$
$$= \frac{I}{E_1^2}\left\{\frac{Z_0\sinh\theta + Za\cosh\theta}{\sqrt{Zb}} + \right.$$
$$\left. + \sqrt{Zb}\frac{(Za\sinh\theta + Z_0\cosh\theta)}{Z_0}\right\}^2 \quad . \quad (63)$$

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As Zb is the only variable, the portion in the outer brackets, as in the preceding cases, is of the same form as equation (89), Chap. I.

If we write $a = Z_0 \sinh\theta + Za\cosh\theta$, and $b = \dfrac{Za\sinh\theta + Z_0\cosh\theta}{Z_0}$

then by (90), Chap. I., we have for a minimum value of $\frac{1}{P}$ and a maximum value of P

$$\frac{Z_0\sinh\theta + Za\cosh\theta}{\sqrt{Zb}} = \sqrt{Zb}\,\frac{(Za\sinh\theta + Z_0\cosh\theta)}{Z_0} \qquad (64)$$

Whence $$Zb = Z_0\frac{(Z_0\sinh\theta + Za\cosh\theta)}{Za\sinh\theta + Z_0\cosh\theta} \quad . \quad . \quad . \quad (65)$$

Now the expression on the right-hand side of (65) is the same as (15), Chap. III., and it is therefore the impedance of a line with apparatus joined to its distant end and closed there, as shown in Fig. 16, Chap. III. Consequently the maximum power in Zb is obtained when its impedance is equal to that of the combined line and apparatus to which it is joined. If $Za = Z_0$ the equation reduces to $Zb = Z_0$. Formula (65) gives the best impedance for maximum energy in the receiving apparatus. If, however, the best impedance for the sending apparatus Za when joined to the combined line and Zb be considered, it may be expected in the light of the preceding results to be obtained when

$$Za = Z_0\frac{(Z_0\sinh\theta + Zb\cosh\theta)}{Zb\sinh\theta + Z_0\cosh\theta} \quad . \quad . \quad . \quad (66)$$

and this may be proved in exactly the same way as in the case of (65). In order that Za and Zb may simultaneously have the best impedance when one of them varies at one time we must have Za and $Zb = Z_0$. It is easily seen that $Za = Zb = Z_0$ is a solution, for if $Za = Z_0$ in (65) and $Zb = Z_0$ in (66), then the equations reduce to $Zb = Z_0$ and $Za = Z_0$. In actual practice each end of a circuit is alternately used for sending and receiving, and the solution $Za = Zb = Z_0$ meets this case also. It follows that a circuit must have similar apparatus at both ends if the best result is to be obtained. The power at the receiving end of a line with similar apparatus at both ends is given by (67) as deduced from (28). Since $Z_2 = \dfrac{E_1}{I_2}$ the power in the receiving apparatus Za is $I_2^2 Za = \dfrac{E_1^2 Za}{Z_2^2}$, that is

$$P = I_2^2 Za = \frac{E_1^2}{4e^{2\theta}Za\left[\frac{1}{4}\left(\frac{Za}{Z_0}+\frac{Z_0}{Za}\right)+\frac{1}{2}+e^{-2\theta}\left\{\frac{1}{2}-\frac{1}{4}\left(\frac{Za}{Z_0}+\frac{Z_0}{Za}\right)\right\}\right]^2} \qquad (67)$$

The quantity in the outer brackets is the same whether the current or the power is in question. The quantity containing the variable impedances (including their angles) in the outer brackets, viz. $\frac{1}{4}\left(\frac{Z_0}{Za}+\frac{Za}{Z_0}\right)$ may be written $\frac{1}{4}\left(\frac{Z_0}{Za}e^{j(\phi-\phi_1)}+\frac{Za}{Z_0}e^{-j(\phi-\phi_1)}\right)$ and the effect of varying the modulus is shown in para. 121.

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125. Conclusions regarding Effect of Impedance.—The following conclusions may be drawn from this investigation :—

(*a*) The maximum power in the apparatus at the end of a transmission line is obtained in all cases when the modulus of the impedance is equal to that of the line or the combined line and impedance to which it is joined.

(*b*) The effect of a change of angle when the impedance changes is in all cases to give an improvement in transmission.

For this reason the so-called "terminal loss" may really amount to a gain in some cases as compared with the infinite line condition, *i.e.* the condition when the modulus of the characteristic impedance of the infinite line is equal to that of the apparatus, and the angles are the same in each case. For example, the effect of the standard apparatus on the standard cable is to give a material improvement as compared with the infinite line condition because the angle $(\phi - \phi_1) = 90°$ in the former case, and is 0 in the latter case, whilst at the same time the ratio $\frac{Z_0}{Za}$ is not very far from equality.

The case of loaded lines is of importance. The angle $(\phi - \phi_1)$ approximates to $/45°$ when the standard apparatus is used, since the angle of the apparatus is 45° and a loaded line has a very small angle. Formula (47) then applies. It should be noticed that even in this instance the angle has a more favourable effect than it has in the case of the infinite line. If, however, we adopt as a practical zero the case of the standard apparatus connected to the standard cable, then by comparison the loaded line connected to the standard apparatus has a marked loss. It is in fact the frequent occurrence of this combination which has given rise to the term "terminal loss."

(*c*) It must not be overlooked that the angular effect is calculated for a frequency of 800 cycles per second, whereas the voice frequencies are many and various, and it is not easy to see the precise effect of angular changes on transmission. It appears, however, from the preceding investigation that any angular change in the impedance at the junction of two impedances gives an improvement in transmission, as compared with transmission in the infinite line, so that some improvement on this account may be generally expected, and this is found to be the case. Calculation at one frequency only, however, when the angle is large, does not always agree closely with the results of voice tests.

126. Applications of Formulæ. Calculated and Observed Effects of Terminal Loss.—It is convenient and useful in telephonic transmission for purposes of comparison, to express current losses at the junction of unequal impedances, in terms of the attenuation on an infinite line, which would produce the same current loss. Thus by (4), Chap. II., $\frac{I_2}{I_1} = e^{-\theta}$, that is the current ratio $\frac{I_2}{I_1}$ between two points at a distance l apart is $e^{-\theta}$ where $\theta = \gamma l$ and γ is the attenuation constant per mile. If $\frac{I_2}{I_1} = 0{\cdot}5$, this means that a current in the infinite line is reduced to half its sending-end value when the

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line angle is θ. Similarly if at the sending end or any other point, the current is reduced by half owing to a change of impedance ratio $\frac{Z_0}{Za}$, the result may be expressed as equivalent to an attenuation such that $e^{-\theta_t} = 0{\cdot}5$ where $\theta_t = \log_e 0{\cdot}5$.

Now Fig. 80 shows the effects of a change of impedance at the ends of an infinite line, for all values of $\frac{Z_0}{Za}$ between 1 and 4. The case illustrated is the theoretical curve for a long loaded line connected to the standard apparatus at both ends of the line. The relative formula is (61) with the proviso that $\cos^2(\phi - \phi_1) = \sin^2(\phi - \phi_1)$ as shown in connection with (47). The value θ_t is plotted against $\frac{Z_0}{Za}$. Here θ_t represents the equivalent attenuation of the terminal loss, and Z_0 and Za are the moduli of these impedances at 800 cycles per second. The equation plotted is

$$\theta_t = \log_e \tfrac{1}{2}\left(\sqrt{\frac{Z_0^2}{Za^2} + \frac{Za^2}{Z_0^2}}\,/\tan^{-1}\phi_2 + 2\right). \quad . \quad (68)$$

It will be seen by Fig. 78 that $\tan\phi_2 = \dfrac{\frac{Z_0}{Za} - \frac{Za}{Z_0}}{\frac{Z_0}{Za} + \frac{Za}{Z_0}}$, if $\cos(\phi - \phi_1)$ $= \sin(\phi - \phi_1)$ and $\sinh\theta = \cosh\theta$.

In actual practice, however, the infinite line condition here taken as the standard of reference is not the ultimate one, but instead, the combination of the standard apparatus with the standard cable is selected as a standard of reference. The standard cable has a characteristic impedance of $571\angle 43° 16'$ and an attenuation constant of 0·106. The theoretical terminal loss between the standard cable and any loaded line, as determined by comparative speech tests, made by the standard apparatus (which has an impedance on zero loop of $425\angle 45°$ ohms), may be determined as follows, on the assumption that the same attenuation is produced by the voice as is calculated for a fixed frequency. In this case 800 cycles per second is the frequency adopted for the calculations. Thus, it is seen by Example 2, Case (*a*), at the end of the chapter, that the effect of the joint impedance of the standard apparatus and standard cable is to give a negative loss, *i.e.* a gain in transmission amounting to $\theta_t = -0{\cdot}6216$ as compared with transmission on the infinite line, whereas by Fig. 80 if a loaded line having an impedance whose modulus is four times that of the standard apparatus be taken, there is an actual loss amounting to 0·35 nearly, as compared with transmission on the infinite line. The infinite line supposed in both cases is the hypothetical one whose impedance is $425\angle 45°$. The total loss of the loaded line as compared with the standard cable, when the standard apparatus is used is therefore $0{\cdot}6216 + 0{\cdot}35 = 0{\cdot}9716$. If this quantity be divided by 0·106 we obtain the number of miles of standard cable which would produce the same attenuation, *i.e.* $\frac{0{\cdot}9716}{0{\cdot}106} = 9{\cdot}16$. Now Fig. 83 shows the

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results of actual speech tests of loaded lines by comparison with the

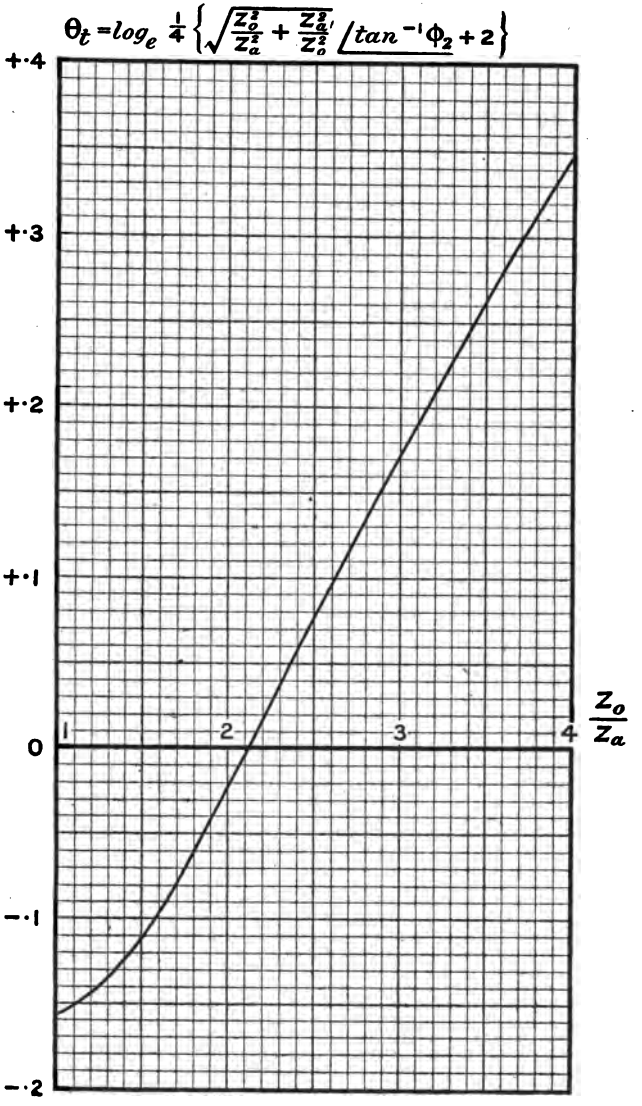

FIG. 80.—Theoretical Terminal Transmission Loss and Gain: standard apparatus connected to long loaded lines at both ends.

standard cable (see Chap. XI. for details of standard cable tests), and it is interesting to compare the results. For the ratio

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$\frac{Z_0}{Za} = 4$, the terminal loss was found by voice tests to be 7·7 miles of standard cable. (See also an experimental curve give by Mr. B. S. Cohen in an article on "Standard Cable Measurements" in the *Post Office Engineers' Journal* for April, 1913, p. 13.) Now the experimental result does not vary largely from the theoretical one, and from this it is evident that the theoretical improvement due to the change of angle is largely realized in practice. If this were not so the terminal loss on the standard cable, if the angle were neglected, would be a positive one instead of a negative one, and the total terminal loss would be very different to that actually observed. The total theoretical loss for the case of a loaded line compared by the standard apparatus with the standard cable is

$$\frac{0{\cdot}6216 + \theta_t}{0{\cdot}106} \text{ miles of standard cable} \quad . \quad (69)$$

where θ_t is the value found from Fig. 80.

127. Transition Loss.—If two long circuits of equal impedance are connected to the ends of a circuit of different impedance, the transmission losses due to the unequal impedances are known as transition losses. If the change of angle in passing from one impedance to another is 45°, Fig. 80 applies to this case also. This is in fact approximately the case of long lengths of the standard cable joined to each end of a long loaded cable. A practical application is provided in the case of a loaded cable having an impedance ratio of 2 : 1 as shown in the curve (Fig. 80). It is found experimentally that when long lengths of the standard cable (10 miles usually suffices) are joined to such a loaded line the terminal loss practically disappears; and it will be seen by the curve that such should be the case. Also, as the standard apparatus is connected to the standard line in all conditions of the comparison, and this combination is the practical zero, no loss occurs at the point of junction of line and apparatus. This test in these conditions provides a means of determining the attenuation due to the cable apart from terminal loss.

In cases where only one junction of unequal impedance requires to be taken into account, half the value shown in Fig. 80 for the same value of $\frac{Z_0}{Za}$ will give an approximately correct result.

For the case where the lines have no angle, or only a small angle (loaded lines), or where the lines Za and Z_0 have the same angle, Fig. 81 has been prepared. This experimental curve is taken from an article by Mr. B. S. Cohen on "Standard Cable Measurements," in the *Post Office Electrical Engineers' Journal* for April, 1913. It is the result of speech tests. The dotted line curve on the same graph shows the losses arrived at by calculation by the author from formula (60) at 800 cycles per second.

128. Terminal Loss with Different Lengths of Local Line in Circuit with the Standard Common Battery Instruments.—Fig. 80 gives the theoretical terminal loss for the case of the standard apparatus joined direct to lines of different impedances. In practice, however, some length of subscriber's line is generally in circuit with the

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standard instrument (see Fig. 135, Chap. XI.). The curves shown in Figs 82 and 83 have been plotted to show the theoretical terminal loss when the standard circuit has different lengths of local line, consisting of standard cable, in series with it. The terminal loss at a frequency of 800 cycles per second has been calculated for the following cases, viz.—

The standard apparatus with all lengths of standard cable from 0 to 6 miles in its local circuit, is connected in turn to both ends of three long trunk circuits consisting of

(*A*) A loaded cable having characteristic impedance of 1700 ohms and no angle.

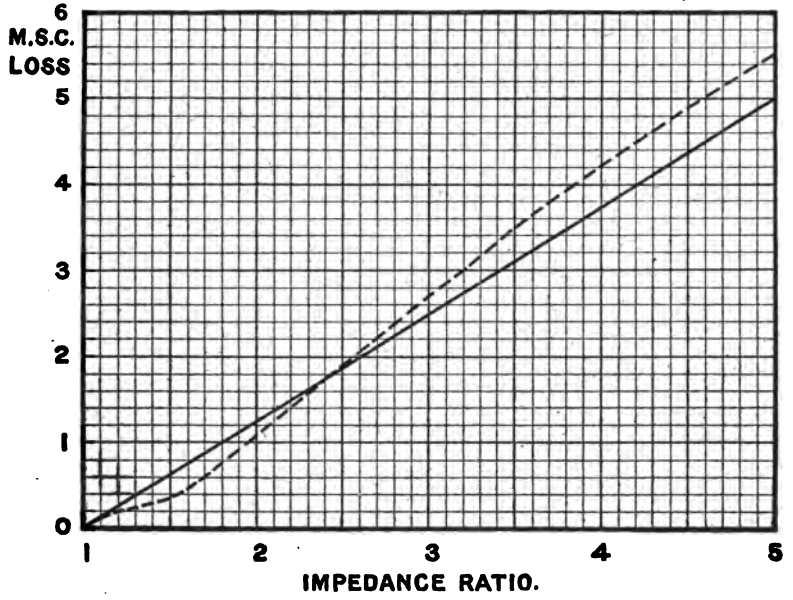

FIG. 81.—Transition Loss—long line (Z_0) terminated by two long lines (Za). The impedances Z_0 and Za have the same angle.

(*B*) A loaded cable having characteristic impedance of 850 ohms and no angle.

(*C*) A long length of unloaded standard cable.

Referring first to C, Fig. 82, at the foot of the graph. This curve shows that when standard apparatus is joined to a long length of standard cable the gain with zero loop (which is the case worked out in Example 2 (*a*)) is a maximum for any length of local line, and that the curve approaches more and more nearly to the zero line in the Fig., or line of no loss, as the length of subscriber's line is increased. In the case of the line having an impedance $Z_0 = 1700$ ohms, the condition is reversed. Here the loss as compared with the line of no loss is greater on zero loop than at any other point on the curve. In the case of the line with an impedance of 850 ohms

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there is very little loss with any length of local line as compared with the infinite line.

The theoretical terminal loss on a line having an impedance $Z_0 = 1700$ ohms at a frequency of 800 cycles per second, by comparison with transmission on the standard cable when the standard apparatus is used in both cases, is the sum of the vertical lengths

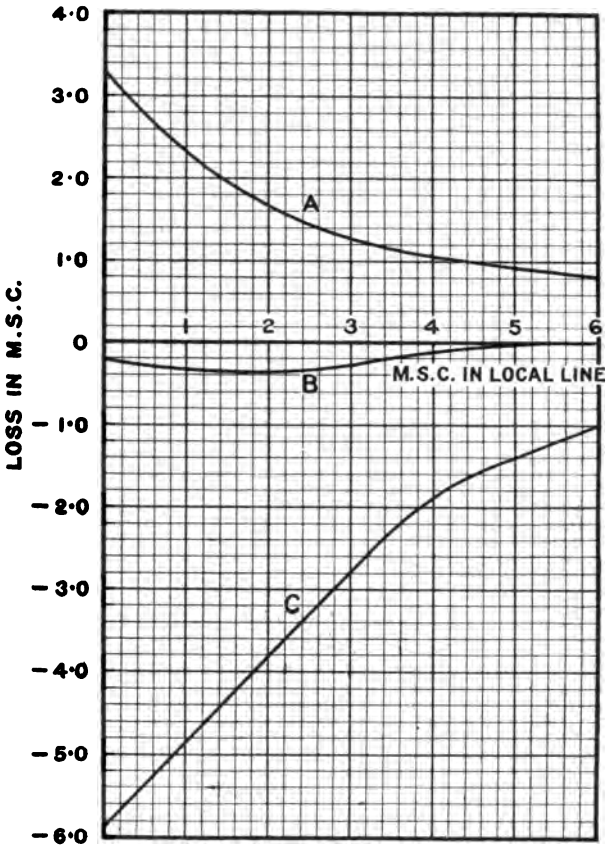

FIG. 82.—Theoretical Terminal Transmission Loss and Gain. Standard apparatus with local lines of different lengths, at the end of three long lines of different types.

between A and C for any given point on the abscissæ or horizontal line, and similarly for the lengths B, C.

A graph of these values, which is the theoretical value of the terminal loss, is given in Fig. 83, and dotted curves of observed results of speech tests are added for comparison. It will be seen that the general character of curves is similar in the two cases, although there is a maximum difference of 1·3 mls. of standard cable between

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the observed and calculated results. When, however, it is remembered that the calculated results are largely affected by the magnitude of the angles of the line and apparatus as calculated for one fre-

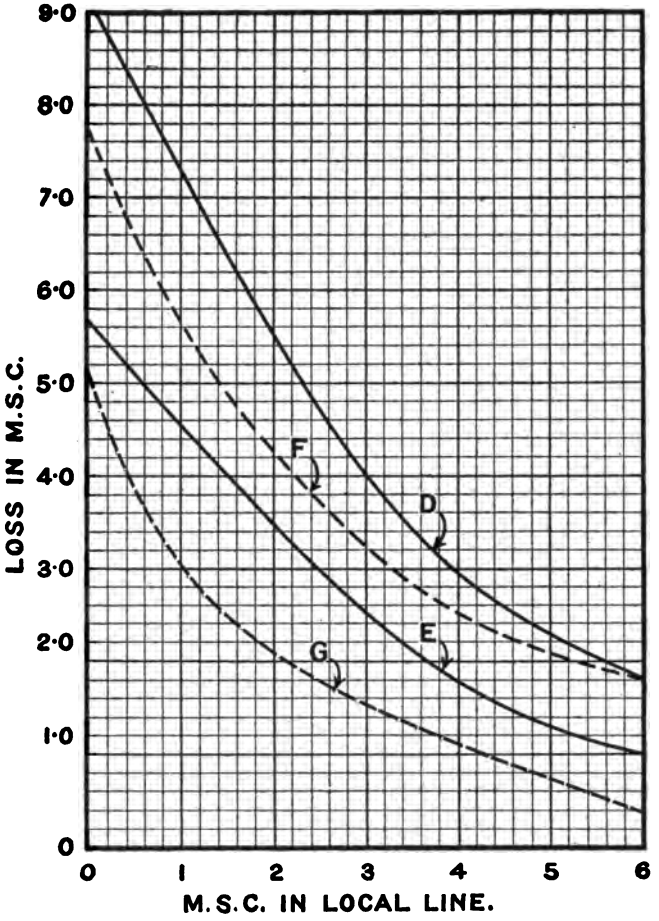

FIG. 83.—Terminal Transmission Loss at both ends of long loaded lines, by comparison with the standard cable, when the standard apparatus is used. Solid lines, calculated. Dotted lines, speaking test. D and F=Heavy loading. E and G=Light loading.

quency, and that on the other hand the voice involves a large number of frequencies, the difference is not surprising. As a matter of fact, standard cable curves of observed speech results also show minor differences when taken at different times.

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Interpolation of Terminal Loss Curves

If the upper curve of observed results in Fig. 83 is taken as a guide, a curve for a line of any other impedance compared with the standard apparatus and line may be approximately found as follows, if the terminal loss with zero loop for the required cases is taken from Fig. 144, or is otherwise known. It is assumed that standard cable is used in the subscribers' loop in all cases for this particular curve. From the point where the new value of the terminal loss on zero loop cuts the ordinate in Fig. 83 drop a perpendicular on the upper curve of observed results, and draw the required curve so that it is always this perpendicular distance from the upper curve, the result will be approximately the desired curve. Similarly for any other type of local line, if one curve is known (preferably the upper one) the others can be approximately found. See also the terminal loss curves in Chap. XI.

This approximate method of interpolation has its limits. A reference to Fig. 80 shows that if the ratio of $\frac{Z_0}{Za}$ is less than 1·5, the curve diverges widely from the approximate straight line law which it has above that figure. If, therefore, a line has an impedance of less than 650 ohms when used in connection with the standard apparatus (425 ohms) the interpolation method described is not recommended.

129. Design of Apparatus.—The standard apparatus appears to be admirably designed for use on the standard cable with local zero loop, as the angle $(\phi - \phi_1)$, which is the difference of the apparatus and line angles, is 90°, and the ratio $\frac{Z_0}{Za}$, where Z_0 is the modulus of the standard cable and Za that of the apparatus is $\frac{571}{425} = 1{\cdot}34$. On the whole a reference to (46) and (48) will show that this combination is highly favourable to efficient transmission. On the other hand, when the same apparatus is connected to a loaded line the angle $(\phi - \phi_1)$ is usually only about 45°, and the ratio $\frac{Z_0}{Za}$ may in some cases amount to 4 or more, where Z_0 is the characteristic impedance of the loaded line and Za the standard apparatus. It is evident, therefore, that the standard apparatus is better adapted to unloaded cables than to loaded lines, and when the use of loaded lines is sufficiently general these considerations make it clear on what lines improvement in the design of transmitting apparatus should proceed. It should not, however, be overlooked that the increased terminal loss in connection with loaded lines is accompanied by a considerable reduction in the attenuation constant, and the possibility of a given change producing secondary effects, whether favourable or otherwise, should constantly be borne in mind.

130. Long and Short Transmission Lines.—Formula (28) gives an expression for the receiving-end impedance of a line of any

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length with apparatus at both ends. If, however, θ is sufficiently great $e^{-2\theta}$ is negligible, and the formula reduces to

$$Z_2 = 2Zae^{\theta}\left\{\tfrac{1}{4}\left(\frac{Za}{Z_0}+\frac{Z_0}{Za}\right)+\tfrac{1}{2}\right\} \quad . \quad . \quad . \quad (70)$$

It is recalled that the quantity $2Zae^{\theta}$ is the impedance of an infinite line with apparatus at its ends having the same impedance as the line (see (25), Chap. III.) and the expression in the brackets is by (60) the formula for the terminal loss in an infinite line. Formula (70) is, from (28), the equation for the receiving-end impedance of an infinite line with similar apparatus at both ends. Now when this condition obtains the variation of the attenuation from that due to the infinite line without apparatus is readily found by curves similar to Fig. 80, which apply to any loaded line which is long enough for (60) to hold good. On the other hand, if the line is short, the full formula [No. (28)] must be taken into account in every case and the calculations become laborious. It is, therefore, of importance to know what electrical length of line is required before (60) can be applied. As already stated, this depends on the quantity $e^{-2\theta}$, *i.e.* on the *square* of the attenuation factor.

Before attempting to interpret Fig. 84 it is advisable to read the remarks under "Calculated and observed effects of terminal loss," in this chapter.

Fig. 84 has been prepared to illustrate the changes in the reflection factor which take place with increase in length. If the variation of A in Fig. 84 with length is shown graphically the effect of the complex angle θ is to produce an undulatory curve which gradually reduces to a smooth uniform line when the factor $e^{-2\theta}$ is negligible. In order to see the effect on speech, however, the attenuation factor $e^{-2\beta}$ has been used instead of $e^{-2\theta}$ for the calculation of Fig. 84, assuming that it represents a mean of the undulatory curve suitable for comparison with speech.

It will be evident on reflection that if in formula A we have $Z_0 = Za$ in magnitude and phase, then $A = 1$ and $\log_e 1 = 0$. The loss is then zero, and this is illustrated by the horizontal zero line in the figure. The curve gives in the ordinate the loss in miles of standard cable for the various conductors marked on the graphs, when those conductors have the attenuation lengths shown in the horizontal base line. The graphs of the conductors represented by solid lines show how the loss A varies with the attenuation length of the line, whilst the dotted lines show what the terminal loss amounts to when the variable portion of A disappears and A becomes constant, *i.e.* when $e^{-2\beta l}\left\{\frac{1}{2}-\frac{1}{4}\left(\frac{Z_0}{Za}+\frac{Za}{Z_0}\right)\right\}=0$. The dotted line bears in every case the same number as the variable curve with which it is associated, and it will be seen that in all cases, as the line increases in length, the variable curve approaches the constant one, and it would eventually, of course, have the same value. Now, as before, the standard cable is the arbitrary standard of reference, and it is seen from the curve that the apparent losses, as determined by comparison with the standard cable, would generally be materially less on short lines than on long ones, but would ultimately attain a

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constant value, which is the terminal loss. Lines similar to the standard cable, *i.e.* small gauge underground lines, would in such a case have a relatively small loss. Even in the case of the 100 lbs. cable shown on the graph, the loss is relatively small, the maximum being less than 2 m.s.c. It will be seen that the difference between

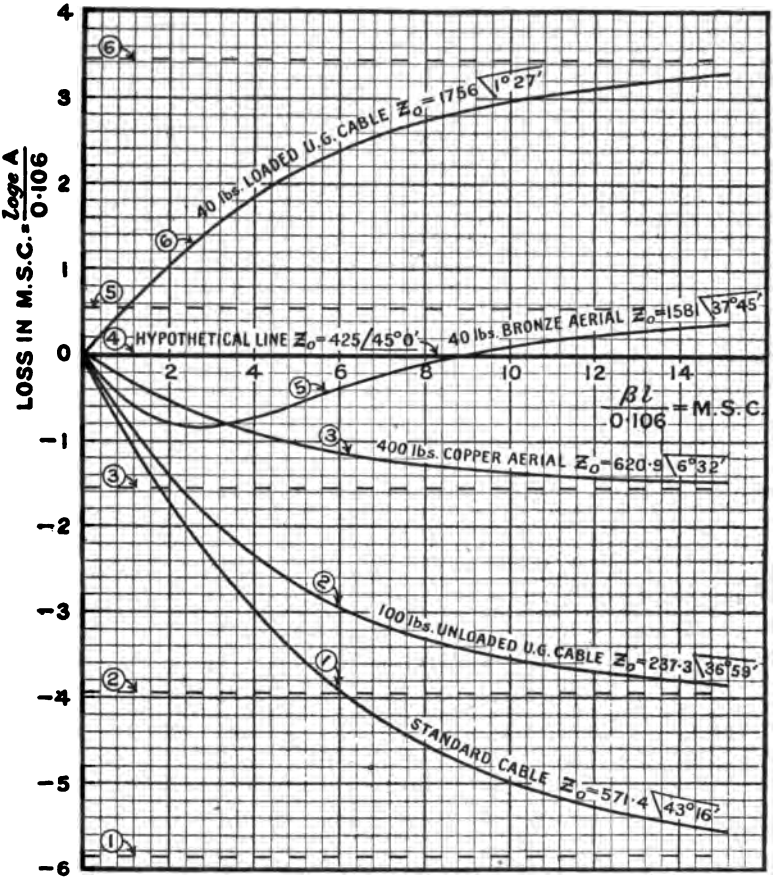
LOSS IN M.S.C. = Loge A / 0·106
40 lbs. LOADED U.G. CABLE Z0 = 1756 \ 1° 27'
HYPOTHETICAL LINE Z0 = 425 / 45° 0'
40 lbs. BRONZE AERIAL Z0 = 158 \ 37° 45'
400 lbs. COPPER AERIAL Z0 = 620·9 \ 6° 32'
100 lbs. UNLOADED U.G. CABLE Z0 = 237·3 \ 36° 59'
STANDARD CABLE Z0 = 571·4 \ 43° 16'
βl / 0·106 = M.S.C.

FIG. 84.—Theoretical Transmission Loss and Gain with Standard Apparatus at both ends of Short Lines. The quantity in the outer brackets in (28) is called A in the ordinate of this figure.

the dotted curve and the one in solid lines for any conductor amounts in no case to more than a mile of standard cable, and is generally much less, when the attenuation length amounts to 10 miles of standard cable. Judging by these curves observed terminal losses on lines of moderate length may be expected to be somewhat less than those calculated from (61).

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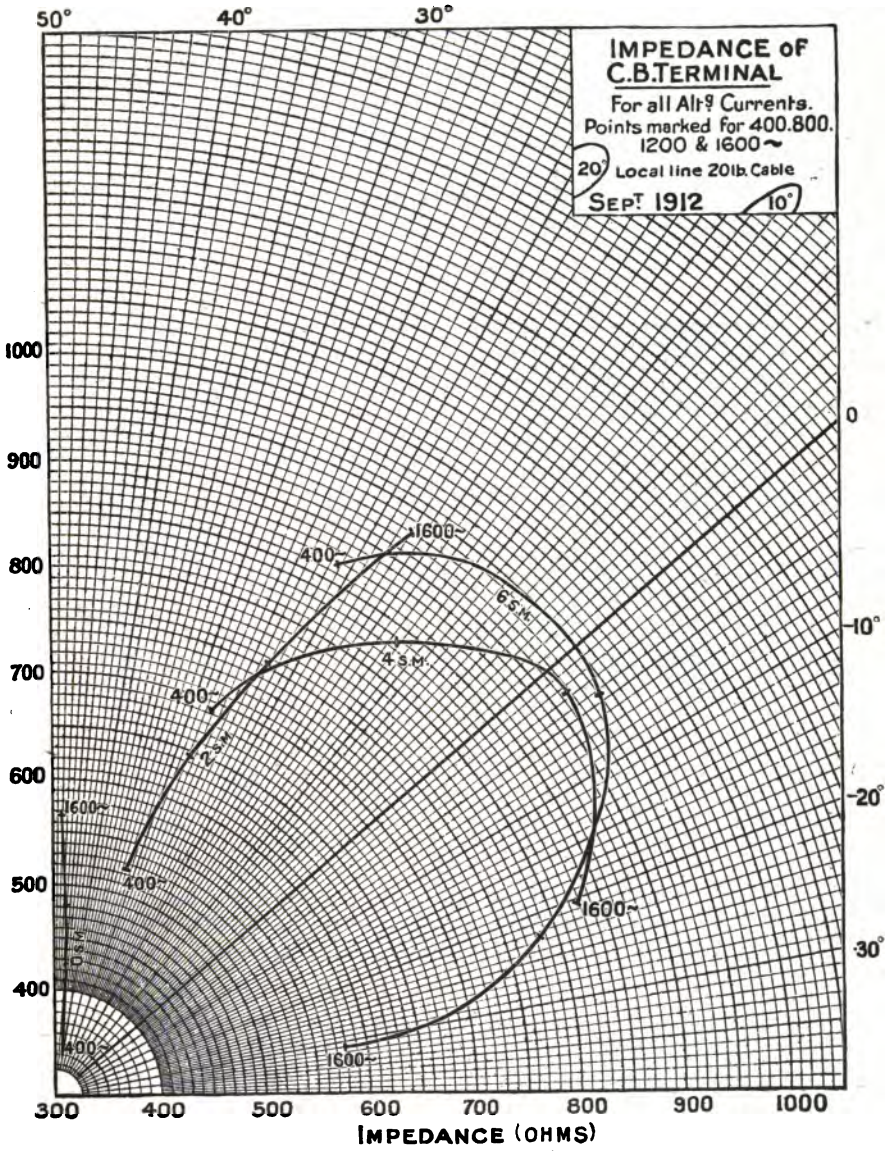

FIG. 85.—The curves show the lengths (S. M.) of standard cable which, when connected to the standard common battery set, have the vector impedances shown in the abscissa and ordinates; the modulus and angle are given separately. The impedance was measured from the terminals of the C.B. set usually connected to the test cable (Fig. 135).

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131. Impedance Curve of Standard Apparatus on Standard Cable.—Fig. 85 gives the impedance of the standard common battery apparatus combined with lengths of line made up of standard cable from 0 to 6 miles in length. It is in use in the Engineer-in-Chief's Laboratory, Post Office, and was published in an article by Cohen and Hill, entitled "Long Distance and Cable Telephony," in *Post Office Electrical Engineers' Journal* for July, 1916.

132. Impedance Curves for Infinite Lines.—Fig. 86 is also in use in the Engineer-in-Chief's Laboratory of the Post Office. It embodies a series of curves which give the loss due to a change of

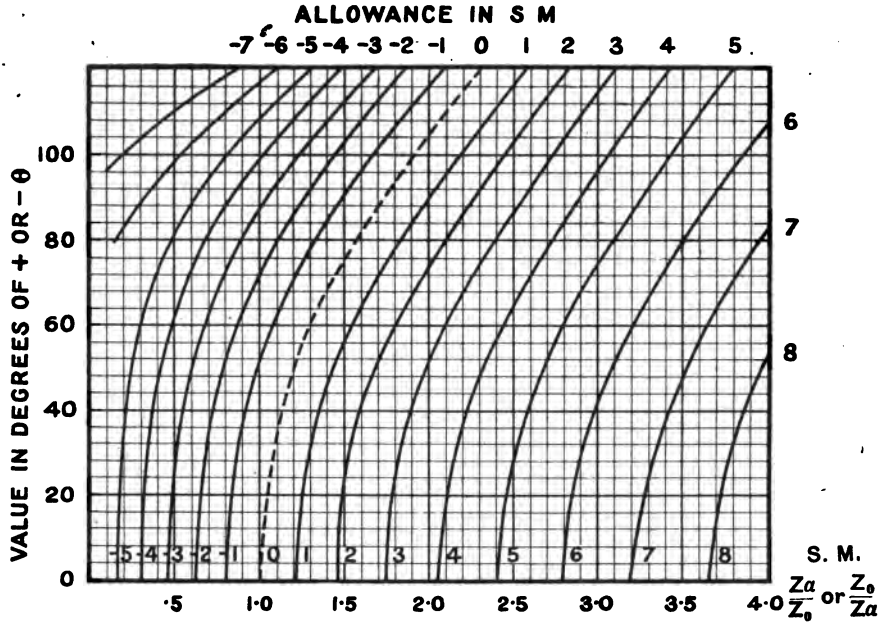

FIG. 86.—Transmission Loss or Gain at the Junction of two Long Lines, or a Long Line and Terminal Apparatus. The modulus is given in the abscissa $\left(\frac{Za}{Z_0} \text{ or } \frac{Z_0}{Za}\right)$, and the angle θ between Z_0 and Za in the ordinate.

impedance at the junction of two infinite lines, or apparatus combined with an infinite line. It is a development of the curves given by Mr. A. J. Aldridge in his paper on the practical application of Telephone Transmission Calculations, read before the Institute of Electrical Engineers on May 30th, 1913. The loss, as determined by this curve, varies in accordance with the direction in which the propagation takes place. In fact, if there is a loss at a junction with the propagation in one direction, there is generally a gain in the other direction. The advantage of this form of curve is that it is applicable to any modulus at any angle. If it be desired to find the loss at both ends of a line, such for example as the loss

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between A and D in Fig. 76, all the lines being infinite in length, proceed as follows. Suppose Z_0 is a long loaded line with an impedance of 1700 ohms and a negligible angle, and that the standard apparatus with an impedance $Za = 425/45°$ is connected at each end. The ratio corresponding to the calculations at the receiving end D is $\frac{Za}{Z_0}$. Therefore this ratio is $\frac{425}{1700} = \frac{1}{4} = 0{\cdot}25$. Refer to this figure at the foot of the graph and trace up in a vertical line until opposite 45°, this gives a loss of −4·7 m.s.c. At the sending end A the direction of the current through the junction is reversed, and we have $\frac{1700}{425} = 4$. Referring to the figure 4 at the foot of the graph and tracing up to 45°, we see that the loss is a little over 8 miles. As a matter of fact, the curve ceases a little before giving the required value, but it is about 8·2 m.s.c. The algebraic sum of these two values is 3·5 m.s.c. The required value may be read direct from Fig. 80, and it will be seen that with the ratio $\frac{Z_0}{Za} = 4$ the terminal loss is 3·3 m.s.c. nearly.

Examples.—(1) Verify by calculation the accuracy of one point in Fig. 80. For this purpose find the value of θ_t when the modulus of $\frac{Z_0}{Za}$ is 4, using formula (47) for the calculation.

We have by (47)

$$e^{\theta_t} = \left[\frac{1}{4}\left\{\sqrt{\left(\frac{Z_0}{Za}\right)^2 + \left(\frac{Za}{Z_0}\right)^2}\,/\tan^{-1}\phi_2 + 2\right\}\right]$$

Substituting the values of $\frac{Z_0}{Za}$ in this formula we obtain

$$e^{\theta_t} = \{\tfrac{1}{4}(\sqrt{16 + \tfrac{1}{16}}/\tan^{-1} 0{\cdot}882 + 2)\}$$

(Note $\tan^{-1}\phi_2 = \tan^{-1}\frac{4-\frac{1}{4}}{4+\frac{1}{4}} = \tan^{-1}\frac{15}{17} = \tan^{-1} 0{\cdot}882 = 41° 25'$.)

$$\begin{aligned} e^{\theta_t} &= \{\tfrac{1}{4}(4{\cdot}0077/41° 25' + 2)\} \\ &= \{\tfrac{1}{4}(4{\cdot}0077\cos 41° 25' + j4{\cdot}0077\sin 41° 25' + 2)\} \\ &= \{\tfrac{1}{4}(3{\cdot}0054 + j2{\cdot}6512 + 2)\} \\ &= \tfrac{1}{4}(5{\cdot}0054 + j2{\cdot}6512) \\ &= 1{\cdot}25135 + j0{\cdot}6628 \\ &= 1{\cdot}4161/ \end{aligned}$$

$$\therefore \theta_t = \log_e 1{\cdot}4161 = \underline{0{\cdot}3478}$$

This agrees with Fig. 80.

(2) *Case* (*a*).—Find the value of θ_t for the terminal loss between the standard apparatus and the standard cable when a very long length of the latter is terminated at each end by the standard apparatus on zero loop

$$Za = 425/45° \text{ ohms}$$

$$Z_0 = 571{\cdot}4/43° 16' \text{ ohms}$$

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Case (*b*).—With the same length of standard cable as before, but with 4 miles of standard cable included in the subscribers' local circuit, find the value of θ_t.

In this case $Za = 838\angle 12° 2'$ ohms.

We have

(*a*) $Z_0 = 571{\cdot}4\angle 43° 16'$ ohms for standard cable.

Also $Z_a = 425\angle 45°$ ohms for standard apparatus.

Hence the terminal loss is by (59)

$$\theta_t = \log_e \frac{(Z_0 + Z_a)^2}{4Z_0Z_a}$$

$$= \log_e \frac{(416{\cdot}0 - j391{\cdot}6 + 300{\cdot}5 + j300{\cdot}5)^2}{4 \times 571{\cdot}4\angle 43° 16' \times 425\angle 45°}$$

$$= \log_e \frac{(716{\cdot}5 - j91{\cdot}1)^2}{971{,}310\angle 1° 44'}$$

$$= \log_e \frac{(722{\cdot}3\angle 7° 15')^2}{971{,}310\angle 1° 44'} = \log_e \frac{521{,}671\angle 14° 30'}{971{,}310\angle 1° 44'}$$

$$= \log_e 0{\cdot}5371\angle 16° 14'$$

i.e. $\theta_t = \log_e 0{\cdot}5371 = -0{\cdot}6216$

(*b*) $Z_0 = 571{\cdot}4\angle 43° 16'$ ohms for standard cable;
and $Z_a = 838\angle 12° 2'$ ohms for standard apparatus at end of 4 mls. of standard cable.

$$\therefore \theta_t = \log_e \frac{(416{\cdot}0 - j391{\cdot}6 + 818{\cdot}7 + j179{\cdot}0)^2}{4 \times 571{\cdot}4\angle 43° 16' \times 838\angle 12° 20'}$$

$$= \log_e \frac{(1234{\cdot}7 - j212{\cdot}6)^2}{1{,}915{,}200\angle 30° 56'}$$

$$= \log_e \frac{1{,}569{,}700\angle 19° 32'}{1{,}915{,}200\angle 30° 56'}$$

$$= \log_e 0{\cdot}8196\angle 11° 24'$$

i.e. $\theta_t = \log_e 0{\cdot}8196 = -0{\cdot}1989$

(3) Calculate the terminal loss for a very long loaded 40 lbs. underground cable having the constants shown in Table II., Chapter VIII., except that the inductance is 200 millihenrys per mile. The cable is terminated at both ends by the standard apparatus on zero loop. The characteristic impedance of the loaded cable is $Z_0 = 1754$ ohms with a negligible angle: use formula (59) for the calculation.

Formula (61) gives the value of the terminal loss θ_t, and this is the same as (59).

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The terminal loss between the loaded line and apparatus is

$$\theta_t = \log_e \frac{(Z_0 + Z_a)^2}{4Z_0Z_a}$$

$$= \log_e \frac{(1754 + 425/45°)^2}{4 \times 1754 \times 425/45°}$$

$$= \log_e \frac{(1754 + 300{\cdot}5 + j300{\cdot}5)^2}{2{,}982{,}000/45°}$$

$$= \log_e \frac{(2076/8° 19')^2}{2{,}982{,}000/45°}$$

$$= \log_e 1{\cdot}445\angle 28° 22'$$

hence $\theta_t = \log_e 1{\cdot}445 = 0{\cdot}3681$

To convert to standard cable units we have $\frac{0{\cdot}3681}{0{\cdot}106} = 3{\cdot}47$

i.e. the terminal loss between the loaded line and apparatus is equivalent to approximately **3½ miles of standard cable.**

If the loaded line be compared with the standard cable by means of the standard apparatus, then with zero loop in the subscribers' line as in Case (*a*), we have by (69)

$\frac{0{\cdot}6216 + 0{\cdot}3681}{0{\cdot}106}$ = terminal loss in miles of standard cable = 9·3 m.s.c.

(The experimental figure for this case is given in Fig. 144; in the upper curve the value is 7·9 miles of standard cable.)

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CHAPTER VIII

THE CONSTANTS OF TELEPHONE CIRCUITS

133. Note on Electrical Constants.—Every telephone circuit has four so-called electrical constants, viz. Electrostatic Capacity, Inductance, Leakance, and Resistance. The use of the term "constant" is a useful convention, but it is of course obvious, even in the case of direct current measurements, that any and all of the "constants" only fulfil that definition in the strict sense in certain well-defined conditions, thus: (*a*) It is evident that for a resistance to remain constant as that term is usually understood, the temperature must remain constant. (*b*) In the case of capacity and inductance the number and proximity of wires on the poles, among other things, affects the measured values. (*c*) As regards leakance, this may have a very wide range of values, and can hardly be said to be a constant.

When the so-called constants are measured at telephonic frequencies, further variations of a complex nature arise. The particular values found at a definite frequency are the "effective" values of the constants, and these values may be, and in some cases are, very different to those found by direct current measurements.

134. Direct Current Resistance.—In Post Office practice telephone conductors are usually designated by their weight in lbs. per mile per wire, when dealing with land lines, and by their weight in lbs. per naut per wire when dealing with submarine cables. In the case of submarine cables it may be remarked that this is a general practice.

The resistance table published by the Post Office in Technical Instruction XIII., relating to the construction of aerial lines, is based on Matthiessen's experiments. Since it was introduced, however, the Engineering Standards Committee (1905) have laid down standards of resistance, and more recently the International Electrotechnical Commission (Berlin, 1913) has formulated international standards, which have been accepted as regards resistance and density by the Institution of Electrical Engineers, and published in their 1916 Wiring Rules.

The following formulæ have been arranged so as to permit of a ready comparison being made between these various standards, particularly from the point of view of the weight per wire per mile. Fuller details are given for the I.E.C. standard than the others.

135. Formulæ based on Matthiessen's Standard as given in Post Office Engineering Dept. Technical Instruction XIII., p. 48 (obtainable from Wyman and Sons).—The resistance in absolute C.G.S. units of a cm. cube of pure hard-drawn copper at 60° F. is 1731. One metre-gramme of pure hard copper has a resistance of 0·1538

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ohm at 60° F. It follows from this that one kilometre-kilogram has a resistance such that R = 153·8 ohms.

$$\therefore R = \frac{153{\cdot}8}{W} \text{ at } 60° \text{ F.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where R is the resistance per kilometre of a uniform copper conductor weighing W kilograms per kilometre.

It is also given that one ohm-mile of pure hard copper has a resistance of 878·5 ohms. Whence

$$R = \frac{878{\cdot}5}{W} \text{ at } 60° \text{ F.} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here R is the resistance per mile of any uniform conductor weighing W lbs. per mile.

The Post Office table of resistances based on these values has been expanded by the author so as to include sizes of wire expressed in the American Brown and Sharpe gauge, and extended so as to include a number of resistances not before given in the table. The author is responsible for the accuracy or otherwise of the additions. The extended table is Table I. in this chapter.

136. Formulæ based on the Engineering Standards Committee's (1905) Definitions.—A wire 1 metre long weighing 1 gramme and having a resistance of 0·1539 ohm at 60° F. (15·6° C.) is taken as the engineering standard for hard-drawn, high-conductivity commercial Cu. For annealed high-conductivity commercial copper the resistance is 0·1508 standard ohm at 60° F. The specific gravity of copper is taken at 8·90 and the weight of 1 cubic cm. at 8·89 grammes. From these definitions the following formulæ are deduced.

For hard-drawn, high-conductivity commercial copper we have

$$R = \frac{153{\cdot}9}{W} \text{ at } 60° \text{ F} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where R is the resistance per kilometre of a wire weighing W kilograms per kilometre.

For annealed high-conductivity commercial copper the formula is as follows:—

$$R = \frac{150{\cdot}8}{W} \text{ at } 60° \text{ F.} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where R is the resistance per kilometre of a uniform conductor weighing W kilograms per kilometre.

Converting these values to resistance per mile and lbs. per mile, we have, for the case of hard-drawn, high-conductivity commercial copper to correspond with (3),

$$R = \frac{878}{W} \text{ at } 60° \text{ F.} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where R is the resistance per mile of a wire weighing W lbs. per mile.

For annealed high-conductivity commercial copper the formula becomes

$$R = \frac{860{\cdot}9}{W} \text{ at } 60° \text{ F} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where R is the resistance per mile of a wire weighing W lbs. per mile.

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The resistance R in ohms per km. of a wire of hard-drawn, high-conductivity commercial Cu having a diameter of d mm. at a temperature of 60° F. is

$$R = \frac{22{\cdot}04}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The resistance R in ohms. per km. of a wire of annealed high-conductivity commercial copper having a diameter of d mm. at a temperature of 60° F. is

$$R = \frac{21{\cdot}6}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The same value as that given in (7) is obtained by taking the resistance in absolute C.G.S. units of a cm. cube of pure hard copper at 60° F. as 1731, in accordance with para. 135. A comparison of (2) and (5) also gives practically the same formula. It will, therefore, be seen that on the basis of volume comparison and weight taken separately, the E.S.C. and Matthiessen's table, as defined in para. 135, give the same result. Page 48 of Technical Instruction, XIII., does not, however, refer to specific gravity, and does not say what constitutes hard-drawn copper. In practice the Post Office resistance table takes the weight of a wire 1 kilometre long and 1 mm. diameter as 7 kg., whereas the definition of 8·89 gm. per cub. cm. leads to a weight of 6·982 kg. per kilometre.

137. Resistance Standard of the International Electrotechnical Commission, Berlin, 1913.—(1) At a temperature of 20° C. the resistance of a wire of standard annealed copper 1 metre in length and of a uniform section of 1 square millimetre is $\frac{1}{58}$ ohm = 0·017241 ohm.

(2) At a temperature of 20° C. the density of standard annealed copper is 8·89 grammes per cubic centimetre.

(3) At a temperature of 20° C. the "constant-mass" temperature coefficient of resistance of standard annealed copper, measured between two potential points rigidly fixed to the wire, is 0·00393 $= \frac{1}{254{\cdot}45}$ per degree C.

The resistance and density involved in this standard are accepted by the Institution of Electrical Engineers and embodied in their 1916 Wiring Rules, as already mentioned.

From the preceding definitions the following values are deduced. The resistance R in ohms per kilometre of a wire having a diameter of d millimetres at a temperature of 20° C. = 68° F. is

$$R = \frac{21{\cdot}952}{d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The resistance R in ohms per km. of a wire weighing W kilogrammes per km. at 20° C. = 68° F. is

$$R = \frac{153{\cdot}3}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the temperature is 60° F. = 15·55° C. this becomes

$$R = \frac{150{\cdot}6}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

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If the temperature is 75° F.=23·88° C. this becomes

$$R = \frac{155{\cdot}6}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The resistance R in ohms per mile at 20° C. of a wire having a diameter of d inches is

$$R = \frac{0{\cdot}05475}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The resistance R in ohms per mile of a wire weighing W lbs. per mile at 20° C. = 68° F. is

$$R = \frac{874{\cdot}9}{W} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The resistance R in ohms per mile of a wire weighing W lbs. per mile at 60° F. is

$$R = \frac{859{\cdot}6}{W} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If the temperature is 75° F. this becomes

$$R = \frac{888{\cdot}2}{W} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The resistance in ohms per naut of a wire weighing W lbs. per naut at 20° C. is

$$R = \frac{1162{\cdot}8}{W} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If the temperature is 60° F. this becomes

$$R = \frac{1142{\cdot}5}{W} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If the temperature is 75° F. this becomes

$$R = \frac{1180{\cdot}5}{W} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

138. Temperature Coefficient of Copper.—The temperature coefficient adopted by the I.E.C. is in agreement with the rule given in the Standardization Rules of the American Institution of Engineers (No. 349), 1916, viz. :—

"The temperature coefficient of copper shall be deduced from the formula

$$\frac{1}{234{\cdot}5 + t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Thus at an initial temperature t=40° C., the temperature coefficient of increase in resistance per degree centigrade is $\frac{1}{274{\cdot}5}$ = 0·00364."

This rule gives a temperature coefficient of increase of 0·00393 for 20° C. as required by the I.E.C. resistance standard. For 60° F. =15·55 C. the American rule gives a coefficient of increase in resistance per degree Fahrenheit of 0·00222, and this value is here adopted.

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NOTE ON TABLE NO. 1.—It is useful to remember that formulæ (3) and (10) give nearly the same result, there is, in fact, only a difference of 0·39 per cent. between them, so that the I.E.C. standard for annealed copper at 20° C. is practically the same as Matthiessen's figure for hard-drawn Cu at 60° F. Table I., which was calculated for the latter, therefore gives results which are sufficiently near to the former for most practical purposes.

139. Iron Wire.—This is occasionally used for minor rural circuits.

The following formulæ are based on the data given on p. 48 of the Post Office Technical Instruction already referred to in this chapter.

The specific resistance, or the resistance in absolute C.G.S. units of a centimetre cube at 60° Fahrenheit is 10616.

One kilometre of pure iron wire weighing W kgs. at 60° F. has a resistance R such that

$$R = \frac{815{\cdot}8}{W} \text{ ohms} \quad . \quad . \quad . \quad . \quad (21)$$

One mile of pure iron wire weighing W lbs. per mile has a resistance R such that at 60° F.

$$R = \frac{4656}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The iron wire used by the Post Office for line wires is commercial iron, having a resistance in absolute C.G.S. units at 60° F. of 12140 per cub. cm.

The resistance R in ohms per mile of a weight of W lbs. per mile of commercial iron at 60° F. is

$$R = \frac{5328}{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

140. Phosphor Bronze Wire.—This wire is used for local circuits. Its resistance at 60° F. is approximately twice that of hard-drawn copper as given in formula (2) (2·07 times). Its diameter is the same as that of copper for the same weight, and its minimum breaking stress at 60° F. is 1·5 times that of hard-drawn copper wire.

141. Aluminium Wire.—This wire has been extensively used as a substitute for copper. The resistance of cast commercial aluminium —99 per cent. pure metal—at 20° C. temperature is 1·685 times that of copper as given in formulæ (9), (10), (13), and (14) for equal cross-sections and lengths of the two metals; but the weight of the copper is then 3·3 times that of aluminium.

For equal resistance the section of the aluminium is 1·685 times that of copper, but the copper still weighs 1·8 times as much as the aluminium. The breaking strain of this Cu is, however, 2·4 times that of aluminium. The temperature coefficient of aluminium is 0·0038 at 0° C. For the harder kinds of aluminium (drawn), suitable for aerial cables, the resistance is increased by 2 per cent. The breaking strain of copper is 2·1 times that of aluminium for this latter type of wire.

142. Effective Resistance—General.—Resistance as measured by direct currents is always accompanied by an energy loss due to

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the heating of the conductor, the amount of which is given by the equation

$$U = I^2Rt \times 0{\cdot}24 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where U is the heat in calories and t the time in seconds.

In the case of alternating currents additional energy losses may arise either in the conductor itself, or external to it due to the action of the magnetic field surrounding the conductor on metallic masses near it. The received current is not in that case determined by Ohm's Law as expressed by the equation $\frac{E}{R} = I$, but by the equation $\frac{E}{Z} = I$, where $Z = \sqrt{r^2 + x^2}$. In this case the apparent resistance r is for the foregoing reasons greater than in the direct current case, and it is called the effective resistance; x is similarly the effective reactance. The increase of resistance in the conductor itself only occurs—to an appreciable extent—with high frequency currents. In the case of double-wire copper conductors, such as those used in telephony and at telephonic frequencies, there is only a small increase over the direct current resistance. In iron conductors, however, the effect is considerable. This arises mainly from the fact that the alternating current produces a magnetic field inside the conductor, as well as outside, and the counter e.m.f. of self-inductance is greatest at the axis of the wire. As a consequence, the current is not distributed uniformly throughout the section of the conductor, the tendency being to produce a concentration of current towards the surface; the greater the conductivity and permeability, and the higher the frequency, the more is the current concentrated towards the surface and the greater is the resistance; this action is known as "skin effect."

In single-wire copper telephone conductors, surrounded by a lead or iron sheath, or near other conductors, the increase of resistance is marked. This arises from the action of the magnetic field upon the sheath or other conductor which leads to energy losses and also from the resistance of the return path (see Table V., Chap. X.).

143. Effective Resistance of Loading Coils.—The magnetic field associated with the conductor is especially great when iron is present, as in this case. The principal phenomena which occur at telephonic frequencies are as follows:—

A magnetic field is created around the wires tested, which induces currents in neighbouring conductors, and in the iron core of the loading coil; energy is absorbed in the conductors and core referred to, and this may be looked upon as equivalent to an increase in the resistance of the loading coil. By Lenz's Law these induced currents would directly oppose the impressed voltage and diminish it. A diminution in volts is equivalent to an increase in resistance in this case. These eddy current losses may be greatly reduced by laminating the iron cores or using bundles of wire instead of a solid core (the losses vary as the sq. of the frequency). In some cases also the wires of the conductor are stranded for the same reason. To still further reduce the losses the windings are separated as far as economy of space allows. Hysteresis losses in

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iron also increase the effective resistance. These losses vary directly as the frequency. Within certain working limits it has been found possible to reduce these losses to a very small or even negligible quantity. A number of formulæ are available for the calculation of the effective resistance at different frequencies (see " Calculation of A.C. Problems," Chap. I., by L. Cohen, published by the McGraw Hill Book Co.).

144. The Effective Resistance of Iron Wire.—Iron is sometimes used for local telephone currents, as before explained. We have here a magnetic material, and the effect of telephonic frequencies is to greatly increase its resistance and decrease its inductance. For small magnetising forces such as those used in telephony, the permeability of iron is found to be approximately constant, and it is known that for ordinary iron telegraph wires a permeability of 100 is a reasonable figure.

The following formula is given by Oliver Heaviside, p. 97, " Electrical Papers," vol. ii. It shows the increase in resistance in any round wire, including iron, due to frequency. Let R be the steady resistance in absolute units per unit length of a round wire of radius r, conductivity κ, permeability μ, and R_1 its effective resistance to sinusoidal currents of frequency $f = \frac{\omega}{2\pi}$. Let also

$$Z = \frac{\mu\omega}{R} = \pi\mu\kappa\omega r^2 = 2\pi^2\mu\kappa r^2 f \quad . \quad . \quad . \quad (25)$$

Then the formula required for R_1 is obtained from

$$\frac{R_1}{R} = \frac{1+\frac{Z^2}{6}\left(1+\frac{Z^2}{2^3\times 10}\left(1+\frac{Z^2}{3^3\times 14}\left(1+\frac{Z^2}{4^3\times 18}\left(1+\frac{Z^2}{5^3\times 22}\left(1+\ldots\right.\right.\right.\right.\right.}{1+\frac{Z^2}{12}\left(1+\frac{Z^2}{3\times 2^2\times 10}\left(1+\frac{Z^2}{4\times 3^2\times 14}\left(1+\frac{Z^2}{5\times 4^2\times 18}\left(1+\ldots\right.\right.\right.\right.\right.} \quad (26)$$

The magnitudes $\frac{R_1}{R}$ for values of Z from $\frac{1}{2}$ to 10 are given by Heaviside. The number of terms in the formula is stated to be suitable up to values of $Z = 10$, beyond that further terms must be taken. A graph showing the relation of Z to $\frac{R_1}{R}$ is given in Fig. 87.

Post Office transmission experiments have shown that values of effective resistance of iron wires calculated on this formula, and also the corresponding one for inductance (32), are fairly consistent with the attenuation constants observed.

145. Inductance—Definition.—The inductance or coefficient of self-induction of a circuit is denoted by the symbol L. The unit is the henry $= 10^9$ absolute units. It is the induction in a circuit when the electromotive force induced in this circuit is 1 volt while the inducing current varies at the rate of 1 ampere per second.

146. Inductance of a Straight Cylindrical Wire.—Where the permeability of the wire is μ and that of the outside medium is unity

$$L = 2l\left(\log_e \frac{2l}{r} - 1 + \frac{\mu}{4}\right) \quad . \quad . \quad . \quad . \quad (27)$$

Here l is the length and r the radius.

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The self-inductance of a length l of straight cylindrical copper wire of radius r is

$$L = 2l\left(\log_e \frac{2l}{r} - \frac{3}{4}\right) \text{ approximately} \quad . \quad . \quad (28)$$

147. Inductance of a Return Circuit.—

$$L = 4l\left(\log_e \frac{D}{r} + \frac{\mu}{4} - \frac{D}{l}\right) \quad . \quad . \quad . \quad (29)$$

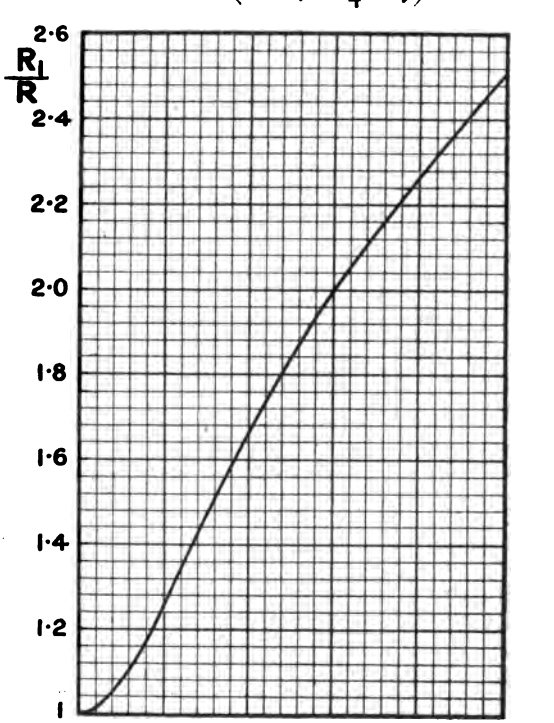

FIG. 87.—Variation of effective resistance with frequency.

If $\mu = 1$ and $\frac{D}{l}$ is negligible, this becomes

$$L = 4l\left(\log_e \frac{D}{r} + \frac{1}{4}\right) \quad . \quad . \quad . \quad . \quad (30)$$

where D is the distance apart of the wires.

(See Formulæ and Tables for the calculation of Mutual and Self-inductance, by E. B. Rosa and L. Cohen, "Bulletin of Bureau of Standards," vol. v., No. 1.)

L is expressed in centimetres in (27) to (30). In the following

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formula (30) is reduced to henrys per mile loop and common logarithms. We then have

$$L = 0{\cdot}001482 \log_{10}\frac{D}{r} + 0{\cdot}000161 \quad . \quad . \quad . \quad (31)$$

In practice, in cases where there are a large number of wires on telephone poles, the value given by this formula may be somewhat increased. Formula (31) may be used for aerial lines or for cables.

148. Inductance of Iron Wires—Variation with Frequency.—When the effective resistance is appreciably greater than the direct current resistance the preceding formulæ for L do not hold good. Oliver Heaviside gives the following formula {which accompanies formula (26)} on p. 97 of "Electrical Papers," vol. ii. Here L_1 is the internal inductance at the frequency involved and $L = \frac{1}{2}\mu$, simply

$$\frac{L_1}{L} = \frac{1+\frac{Z^2}{2^2\times 6}\left(1+\frac{Z^2}{2\times 3^2\times 10}\left(1+\frac{Z^2}{3\times 4^2\times 14}\left(1+\frac{Z^2}{4\times 5^2\times 18}\right(+\dots}{1+\frac{Z^2}{12}\left(1+\frac{Z^2}{3\times 2^2\times 10}\left(1+\frac{Z^2}{4\times 3^2\times 14}\left(1+\frac{Z^2}{5\times 4^2\times 18}\right(+\dots} \quad (32)$$

This formula is general for any frequency.

A graph showing the relation between $\frac{L_1}{L}$ and Z {see (25)} for values of Z between 0 and 10 has been calculated for reference, and this is given in Fig. 88.

149. Leakance.—The dielectric losses in telephone conductors are generally enormously greater for currents of telephonic frequency than for direct currents; for example, the leakance in a gutta-percha insulated cable, at a frequency of $2\pi f = 5000$, may be, and sometimes is, at least 50,000 times as great as the direct current leakage in the same dielectric. The alternating current dielectric loss is generally represented as obeying a law which may be expressed by the following equation:

$$A + B\omega = G \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Where A is the constant leakage, which is independent of frequency, and B is a constant which indicates the magnitude of a loss known as electrostatic or dielectric hysteresis. This loss is analogous to hysteresis; it consists of a loss of energy per cycle, and is proportional to the frequency, the electrostatic capacity, and the square of the voltage. If the dielectric hysteresis component of (33) is very large compared with A, then the loss is such that $\frac{G}{\omega C}$ is a constant quantity for a fixed voltage, and this assumption is generally made in transmission calculations. It should, however, be borne in mind that these calculations are generally made for a fixed frequency ($2\pi f = 5000$).

Steinmetz, in his "Alternating Current Phenomena," p. 145, McGraw Publishing Co., 1900 edition, says it is quite possible that a third energy loss component, proportional to the square of the frequency, and analogous to the loss of power by eddy currents in iron, may exist. In that case there is a close analogy between

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the energy losses in the conductor and those in the dielectric, as will be seen from the following comparison.

Losses in Conductor.—Resistance + hysteresis + eddy current losses.

Losses in Dielectric.—Leakage + dielectric hysteresis + energy loss varying as ω^2.

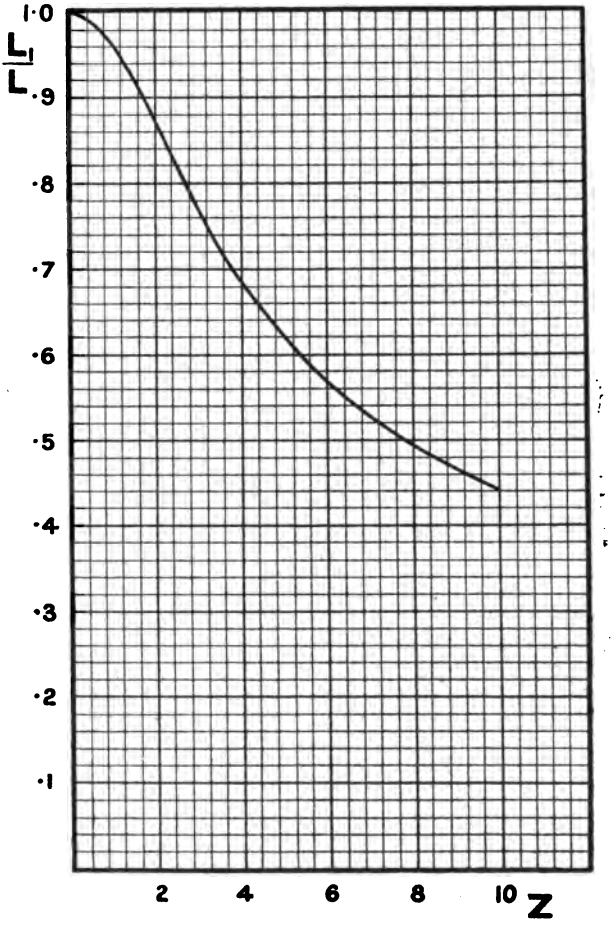

FIG. 88.—Variation of Inductance with Frequency.

In each case the first term is a constant independent of frequency, the second term is proportional simply to the frequency, and the third term is proportional to the square of the frequency.

Experiments on aerial telephone lines show that the leakance in that case varies directly as the frequency approximately. Experiments made by Fleming and Dyke, and reported in a paper on

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"Power Factor and Conductivity," read before the Institute of Electrical Engineers, and given in the *Journal* of the Institute for 1912, show that in by far the greater number of cases, the alternating current conductance is a simple linear function, and of the form of equation (33). It was found, among other things, by the authors named, that dry core telephone paper cable obeys the straight line law, and that A was very small compared with $B\omega$. This has been confirmed by Post Office experiments.

On the other hand, Post Office experiments on gutta-percha insulated submarine cables, given in Chapter X., show that the leakance varies appreciably from the straight line law.

From the practical point of view, both calculation and experiment show that the production of paper insulated, and gutta-percha covered cables, with a low leakance, is both important and possible. Recent improvements in this respect have resulted in the speaking range of submarine cables being increased by 40 per cent. At present a value of $\frac{G}{C} = 20$ can be obtained for both paper insulated and balata-covered cables. The leakance shown in submarine cables with gutta-percha insulation is very much higher. Experimental data are given in Figs. 125 and 126, Chapter X.

150. Electrostatic Capacity or Capacitance—Definition.—The practical unit of capacity is the farad. This unit is one one-thousand millionth of the absolute unit of capacity, or 10^{-9} absolute units. It is the capacity of a condenser which is charged to a potential of 1 volt by one coulomb (10^{-1} absolute units of quantity). The microfarad or one-millionth of a farad is largely used in practice.

151. Capacity of Double-Wire Aerial Line without Iron.—

$$C = \frac{\cdot 0194}{\log_{10} \frac{2D}{d}} \text{ mf. per mile} \quad . \quad . \quad . \quad (34)$$

For proof see "The Propagation of Electrical Currents in Telegraph and Telephone Conductors," by J. A. Fleming, p. 193.

152. Capacity of a Single-Wire Aerial Line without Iron.—

$$C = \frac{0 \cdot 038}{\log_e \frac{4h}{d}} \text{ mf. per mile} \quad . \quad . \quad . \quad . \quad (35)$$

In these formulæ d is the diameter of the wire, D is the distance apart of the wires, and h the height of the wires above the ground. All the quantities must be expressed in terms of the same unit, *i.e.* millimetres or inches. In actual practice, when there are a large number of wires on the poles, the values given by these formulæ require to be increased.

Mr. H. R. Kempe found as a result of experiments on single telegraph wires in actual working conditions on a line carrying a number of wires, that the following formula for a single wire holds good:—

$$C = \frac{0 \cdot 0616}{\log \frac{4h}{d}} \text{ mf. per mile} \quad . \quad . \quad . \quad . \quad (36)$$

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In the single-wire experiments referred to in Table III., Chap. X., it should be noted that there were only four wires on the poles. In view of the fact that the capacity depends on the height of the poles, and that this may vary largely, it is difficult to make a conclusive comparison between calculated and observed figures.

153. Capacity of Underground Cables.—The capacity of telephone pairs in cables containing a large number of closely packed wires depends on the dielectric constant, and a factor depending on the form and dimensions of the cable. The usual course in the case of paper-core air space cables (which have a dielectric constant varying from 1·7 to 1·9) is to design cables on the basis of carefully ascertained experimental data. The subject is dealt with in an article by Mr. S. Hanford, B.Sc., of the Engineer in Chief's Office, G.P.O., on the "Determination of Dimensions and Weight by Means of Curves." See *The Post Office Electrical Engineers' Journal*, for October, 1914.

"The spaces allowed per pair of conductors, corrected to the time of writing, together with the standard capacity at present specified by the Post Office, are shown in the following table :—

PAPER CORE TELEPHONE CABLES.

Weight of conductors and types of cores per mile per hour.	Space per pair (square inches).	Specified capacity wire to wire (microfarads per mile).
6½-lb. twin cable	0·0040	Mean not to exceed 0·075
10 ,, ,,	0·0057	,, ,, ,, 0·075
20 ,, ,,	0·0140	,, ,, ,, 0·065
40 ,, ,,	0·0215	,, ,, ,, 0·065
70 ,, ,,	0·0475	,, ,, ,, 0·065
20-lb. multiple twin cable	0·0140	,, ,, ,, 0·065
40 ,, ,, ,,	0·0275	,, ,, ,, 0·065
70 ,, ,, ,,	0·0475	,, ,, ,, 0·065
100 ,, ,, ,,	0·0680	,, ,, ,, 0·065
150 ,, ,, ,,	0·1020	,, ,, ,, 0·065
200 ,, ,, ,,	0·1360	,, ,, ,, 0·065

Details of the experimental formulæ and data required for the calculation of capacity in telephone cables will be found in "Electric Cables, Their Construction and Cost," by D. Coyle and F. J. O. Howe, published by E. and F. N. Spon, Ltd., 1909, p. 166, etc.

154. Capacity of Phantom Circuits in Paper-Core Telephone Cables.—When cables are made up in the so-called multiple twin formation, *i.e.* when two telephone pairs are twisted to form a "phantom pair," the capacity of these phantom pairs may vary within wide limits. In the London-Liverpool cable (para. 189), the mean increase in capacity of the phantom circuits was 50 per cent. over that of the capacity of the pairs on which the phantoms were superimposed. The range of variation was from 26 to 59 per cent., but the former figure was quite exceptional, seeing that out of ten observed results nine varied between 47 and 59 per cent. The observed mean of 50 per cent. increase in the capacity of the

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phantom circuits, as compared with the sides, is the same as the observed increase in capacity of a single wire as compared with the wire-to-wire capacity of a pair in paper-core telephone cables, *i.e.* two single wires which are taken to make up a pair have (in a large number of cases) each about 1·5 times the capacity of the pair when it is measured wire to wire. It is, however, possible to have considerable variations from these figures.

155. Capacity of a Submarine Cable.—A submerged single wire core may be represented for capacity calculations by two long co-axial circular conducting cylinders separated by the dielectric, the inner cylinder being the conductor, and the outer the water. The formula for the capacity to earth of a unit length of such a cylinder is

$$C = \frac{1}{2K \log_e \frac{r_2}{r_1}} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

Where r_2 is the radius of the outer cylinder and r_1 of the inner one and $K = \frac{1}{\text{dielectric constant}}$. This formula for the case where $K = 1$ is proved in Poynting and Thomson's "Electricity and Magnetism," Part I., p. 83 (published by Griffin & Co., Ltd.).

If (37) is expressed in absolute units per cm. it may be converted to microfarads per kilometre by multiplying by $\frac{10^5}{9 \times 10^5}$. We then have

$$C = \frac{1}{2K \times 9 \times \log_e \frac{r_2}{r_1}} \text{ mf. per km.} \quad . \quad . \quad (38)$$

Converting to common logs and taking the value of the dielectric constant for gutta-percha at 3·6 we obtain

$$C = \frac{3{\cdot}6 \times 0{\cdot}4343}{18 \log_{10} \frac{r_2}{r_1}} = \frac{0{\cdot}08686}{\log_{10} \frac{r_2}{r_1}} \text{ mf. per km. for g.-p.} \quad (39)$$

To convert to nauts we have

$$C = \frac{0{\cdot}08686}{\log_{10} \frac{r_2}{r_1}} \times 1{\cdot}85 = \frac{0{\cdot}1607}{\log_{10} \frac{r_2}{r_1}} \text{ mf. per naut}$$

But $\frac{D_0}{d} = \frac{2r_2}{2r_1}$, where D_0 is the outer diameter of the g.-p. and d the diameter of the copper conductor.

Therefore the wire to earth capacity of one naut of g.-p. cable is

$$C = \frac{0{\cdot}1607}{\log_{10} \frac{D}{d}} \text{ mf.} = \frac{0{\cdot}16}{\log_{10} \frac{D_0}{d}} \text{ approx.} \quad . \quad . \quad (40)$$

The value 3·6 for the dielectric constant is sometimes given; but the figure varies in practice between three and four, depending

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on the treatment and composition of the gutta-percha. The effective diameter d to be used for capacity calculations, in the case of round stranded wires, in order to allow for the fact that the external diameter is not that of a perfect cylinder, is a little less than the outer diameter of the strand; a reduction of 5 per cent. on this diameter is a probable mean figure for practical purposes. The capacity of a double-wire phantom circuit is generally double that of the two-wire side circuit in a submarine cable, and a pair wire to wire has half the capacity of a single wire to earth.

156. Temperature for Submarine Cable Measurements.—Factory measurements of insulation, resistance, etc., are generally made at a temperature of 75° F. = 23·88° C. The insulation of gutta-percha varies considerably with temperature.

157. Electrification and Absorption in Submarine Cables.—The measurement of leakance at telephonic frequencies is dealt with in paragraph 149. When, however, the apparent direct current insulation of a submarine gutta-percha or balata insulated cable immersed in water is measured, the observed reading varies rapidly with time. When the testing battery—say 150 volts—is first applied, there is a large rush of current into the cable owing to its capacity, but a diminishing charge continues to flow into the cable for a considerable time if the battery is continuously applied, gradually becoming less with lapse of time. This phenomenon is known as electrification, and the insulation usually recorded as the insulation of the cable in Post Office practice is the apparent insulation after a charge of one minute. Readings are, however, taken at the end of every minute for, say, twenty minutes. If the insulated cable is allowed to fully charge for that time, and is then earthed, a reverse phenomenon occurs, and a current due to the "absorption" flows out. If the recorded curves of charge and discharge with time are smooth and regular, it may be regarded as a sign of the good condition of the dielectric. The dielectric resistance decreases as the temperature increases. A curve connecting insulation with temperature is usually provided by the manufacturer of modern telephone cables. The usual insulation required for gutta-percha is 400 to 2000 megohms per naut. In the case of balata the insulation may be much less as measured by direct current, and much more favourable as computed from the ratio $\frac{G}{\omega C}$ at telephonic frequencies (see para. 149).

158. The Attenuation Constant—The Most General Case and the Ideal Case.—The most general form of the attenuation constant is that proved in Appendix I., which is

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+\tfrac{1}{2}(RG-\omega^2CL)} \quad (41)$$

This is applicable to any telephone circuit whatever with uniformly distributed constants; but even if in any case all the constants require to be taken fully into account, as provided for in (41), it is easier to use the complex attenuation constant to obtain β as shown in Chapter VI., Example 1, by the use of vectors.

The theoretical minimum value of β is

$$\beta_m = \sqrt{RG} \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

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and this is attained when $LG = CR$, as proved in Appendix V. In ordinary unloaded telephone circuits LG is always less than CR, and there are two possible and obvious ways of increasing LG, *i.e.* either by increasing L or G. Now (42) not only gives the minimum value of β, but when this equality is arrived at, it is seen by inspection that the frequency factor ω does not come into equation (42), and this means that the circuit is theoretically distortionless, *i.e.* all frequencies are equally attenuated, which, of course, indicates that the line transmits speech without distortion. It is clearly more advantageous to attain the conditions of (42) by increasing L in preference to increasing G, because an increase in L does not necessarily affect R or G, although these constants determine the *magnitude* of the minimum β, whereas an increase of G, although it brings about a distortionless condition, may actually increase β. An example will assist in making this clear. To simplify the matter, a circuit will be selected which might in actual practice, on the assumption of very low insulation, attain the conditions of (42) without any artificial leak. An unloaded 800 lbs. aerial circuit will meet the case. We have by Table II.—

$R = 2{\cdot}2$ ohms per mile loop.
$L = 0{\cdot}00322$ henry per mile loop.
$C = 0{\cdot}0099$ microfarad per mile loop.
$G = 1$ micromho per mile loop.

With these constants the attenuation constant is

$$\beta = 0{\cdot}00221 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

If G is varied to attain the equality $LG = CR$, we have

$$0{\cdot}00322 \times G = 0{\cdot}0099 \times 10^{-6} \times 2{\cdot}2.$$

Whence $G = 6{\cdot}76$ micromhos or $\frac{1}{G} = \frac{1}{6{\cdot}76} \times 10^6$ ohms $= 148{,}000$ ohms per mile. The attenuation constant in that case by (36) becomes $\beta = 10^{-8}\sqrt{2{\cdot}2 \times 6{\cdot}76} = 0{\cdot}0039$, which is considerably greater than (43), and consequently the volume of speech would be much less than if the normal insulation of one megohm per mile were used, although the distortionless condition would be attained.

159. Volume and Clearness of Speech in Aerial Lines and Underground Cables.—The fact that the circuit is rendered distortionless would be an undoubted advantage if the clearness of speech on an aerial circuit of large gauge left anything to be desired; but, as a matter of fact, speech on such unloaded wires is quite clear. This is corroborated by the details of tests on a 300-lbs. aerial circuit given in Chap. X., where it may be observed that there is very little change in the circuit constants when tested at frequencies varying from $\omega = 3000$ to $\omega = 7000$. In unloaded underground circuits, however, there is considerably more distortion, and the distortionless circuit would undoubtedly be of great advantage in that case, if it were possible to attain it without loss of volume. In practical telephony, however, a sufficient intensity or volume is absolutely essential in order to permit of conversation being carried on in the normal noises of open rooms. Experience

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shows that speech can be carried on even with a fairly large amount of distortion, whereas distortionless speech of relatively small volume might be inaudible. As a matter of fact, the method of attaining the condition of (42) by an increase of G is never adopted, because the same theoretical advantage can be obtained and the attenuation constant diminished by loading.

160. The Attenuation Formula for a Loaded Circuit when ωL is Great in Proportion to R, and ωC is great in Proportion to G.—We then have by Appendix VI.—

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad (44)$$

This condition is brought about by artificially increasing L, and this is fully dealt with in Chap. IX.

It will be observed that (44) is a distortionless formula, seeing that the frequency factor ω does not appear in it, and the conditions in which it may be used are indicated in the preceding heading. In actual practice both R and G vary with frequency. It should again be borne in mind that all the constants in these formulæ are effective values, *i.e.* the particular values of the constants for some definite frequency, as already pointed out.

161. Minimum Value of Formula (44).—On the assumption that only L varies in the expression on the right-hand side of (44) the minimum may be found as follows :—

If $\frac{R}{2}\sqrt{C}$ be called a, and $\frac{G}{2\sqrt{C}}$ be called b, (44) becomes

$$\beta = \frac{a}{\sqrt{L}} + b\sqrt{L} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

This equation is of the same form as (89), Chap. I., and therefore by (90), Chap. I., the minimum value is obtained when $\frac{a}{\sqrt{L}} = b\sqrt{L}$, or

$$\frac{R}{2}\sqrt{\frac{C}{L}} = \frac{G}{2}\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

If each side of this equation is multiplied by $2\sqrt{CL}$, we have $CR = LG$, which is the same condition as that required to attain (42). In this case, however, formula (44) applies not only when $LG = CR$, but in any case when the conditions stated immediately before (44) are fulfilled.

When the conditions of equation (46) are fulfilled all the attenuation loss is energy loss, *i.e.* it is the sum of the losses arising from the heating of the conductor (I^2R), and direct loss of current by leakage or absorption, and such losses can only be avoided (with a constant value of current) by reducing R and G. (44) may be written

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2}\sqrt{\frac{L}{C}} \times \frac{L}{C} \times \frac{C}{L} = \frac{R}{2}\sqrt{\frac{C}{L}} + \frac{G}{2C}L\sqrt{\frac{C}{L}}$$

that is

$$\beta = \frac{R + \frac{G}{C}L}{2}\sqrt{\frac{C}{L}} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

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Now R is an effective resistance due to the line resistance per mile + any resistance added by the inductance used to load, and this added effective resistance is usually proportional to the inductance. So that if R_1 is the resistance added by one henry L_1, then $\frac{R_1}{L_1}L$ is the resistance for any other value L, and (47), in this case, may be written

$$\beta = \frac{R + \frac{R_1}{L_1}L + \frac{G}{C}L}{2}\sqrt{\frac{C}{L}}$$

that is

$$\beta = \frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L}{2}\sqrt{\frac{C}{L}} \quad . \quad . \quad . \quad . \quad (48)$$

In order to obtain the minimum value of β when written in this form (48) may be rearranged as follows as a first step:

$$= \left\{\frac{R}{2\sqrt{L}} + \frac{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)}{2}\sqrt{L}\right\}\sqrt{C}$$

Now R, $\frac{R_1}{L_1}$, and $\frac{G}{C}$ are constants, and if $\frac{R}{2}$ is called a, and $\frac{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)}{2}$ is called b, (42) becomes

$$\beta = \left(\frac{a}{\sqrt{L}} + b\sqrt{L}\right)\sqrt{C} \quad . \quad . \quad . \quad . \quad (49)$$

This equation is of the same form as (89), Chap. I., if $\sqrt{L}$ is the unknown variable. The minimum value of β by (90), Chap. I., is then obtained, when

$$\frac{a}{\sqrt{L}} = b\sqrt{L} \quad \text{or} \quad R = \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L \quad . \quad . \quad (50)$$

Notice that $\sqrt{C}$ in (49) is constant, and therefore does not affect this result.

When $R = \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L$ then (48) may be written $\frac{2R}{2}\sqrt{\frac{C}{L}}$ and

$$\beta_m = \sqrt{\frac{R^2C}{L}} = \sqrt{\frac{RC\left(\frac{R_1}{L_1} + \frac{G}{C}\right)L}{L}}$$

Whence

$$\beta_m = \sqrt{CR\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} \quad \text{or} \quad R\sqrt{\frac{C}{L}} \quad . \quad . \quad (51)$$

If in (44) $\sqrt{\frac{L}{C}}$ be called Z_0 the formula becomes

$$\beta = \frac{R}{2Z_0} + \frac{GZ_0}{2} \quad . \quad . \quad . \quad . \quad . \quad (52)$$

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162. Note on the Attenuation Constant of Phantom Circuits.—If the effective constants R, L, G, and C are known, the formulæ for the attenuation constants and characteristic impedance apply equally to both types of circuits (side and phantom). There is not a necessary fixed ratio between the efficiency of phantom circuits and side circuits.

163. Attenuation Constant for Loaded Open Lines.—In this case the natural inductance of the line cannot be neglected, and this is taken into account by adding L_n, the natural inductance of the line, to the artificial inductance L in formula (48). We will, however, write L_c for the inductance of the coil per mile in this case. (48) then becomes

$$\beta = \frac{R + \frac{R_1}{L_1}L_c + \frac{G}{C}(L_c + L_n)}{2}\sqrt{\frac{C}{L_c + L_n}}$$

($\frac{R_1}{L_1}L_c$ is the effective resistance per mile added by the coil and the natural inductance does not affect it);

that is
$$\beta = \frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L_c + \frac{G}{C}L_n}{2}\sqrt{\frac{C}{L_c + L_n}}$$

$$= \frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)(L_c + L_n) - \frac{R_1}{L_1}L_n}{2}\sqrt{\frac{C}{L_c + L_n}}$$

Next write L for $L_c + L_n$ then we have

$$\beta = \frac{R - \frac{R_1}{L_1}L_n + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L}{2}\sqrt{\frac{C}{L}} \quad . \quad . \quad . \quad (53)$$

$$= \left\{\frac{\left(R - \frac{R_1}{L_1}L_n\right)}{2\sqrt{L}} + \frac{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)\sqrt{L}}{2}\right\}\sqrt{C} \quad . \quad . \quad (54)$$

Now all the quantities except L are constants. If $\frac{\left(R - \frac{R_1}{L_1}L_n\right)}{2} = A$ and $\frac{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)}{2} = B$, then (54) becomes $\beta = \left(\frac{A}{\sqrt{L}} + B\sqrt{L}\right)\sqrt{C}$ and this is of the same form as (89), Chap. I., and therefore by (90), Chap. I., for minimum β we have $\frac{A}{\sqrt{L}} = B\sqrt{L}$ or $A = BL$

$$\therefore R - \frac{R_1}{L_1}L_n = \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L \text{ or } L = \frac{R - \frac{R_1}{L_1}L_n}{\frac{R_1}{L_1} + \frac{G}{C}} \quad . \quad . \quad . \quad (55)$$

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Therefore (53) may be written

$$\beta_m = \left(\frac{R_1}{L_1} + \frac{G}{C}\right) L \sqrt{\frac{C}{L}} = \sqrt{\left(\frac{R_1}{L_1} \quad \frac{G}{C}\right)^2 LC}$$

Substitute the value of L from (55), and this then becomes

$$\beta_m = \sqrt{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)^2 \frac{\left(R - \frac{R_1}{L_1} L_n\right)}{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} C}$$

or

$$\beta_m = \sqrt{C\left(R - \frac{R_1}{L_1} L_n\right)\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} \quad . \quad . \quad . \quad (56)$$

164. Attenuation Constant for Unloaded High Resistance Cables.—If G and L are each o, formula (41) reduces to

$$\beta = \sqrt{\tfrac{1}{2}\omega CR} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

In the case of telephone circuits this is only true for wires of small gauge; the formula is easily deduced by putting o for G and for L in (41).

In cases of doubt the accuracy of the formula for any particular case may be verified without much trouble by the use of (2), Appendix V., if the values of $\frac{G}{C}$ and ωL applying to the unloaded cable are known, as is usually the case. The formula is

$$\beta = \sqrt{\tfrac{1}{2}\omega CR\left\{\sqrt{\left(1 + \frac{\omega^2 L^2}{R^2}\right)\left(1 + \frac{G^2}{\omega^2 C^2}\right)} + \frac{G}{\omega C} - \frac{\omega L}{R}\right\}} \quad (58)$$

The following details respecting underground air-space paper cables are extracted from Table II:—

$L = 0{\cdot}001$ henry per mile loop.
$G = 1 \times 10^{-6}$ mhos ,, ,,
$C = 0{\cdot}065 \times 10^{-6}$ farads per mile loop.

$$\therefore \frac{G}{C} = \frac{10^{-6}}{0{\cdot}065 \times 10^{-6}} = \frac{1000}{65} = 15.$$

This is not an absolutely rigid figure, and as a matter of fact 20 is usually taken to allow for contingencies.

$$\omega^2 = (5000)^2 = 25 \times 10^6$$

Inserting these values in (58) we have

$$\beta = \sqrt{\tfrac{1}{2}\omega CR\left\{\sqrt{\left(1 + \frac{25 \times 10^6 \times 10^{-6}}{R^2}\right)\left(1 + \frac{20^2}{25 \times 10^6}\right)} + \frac{20}{5000} - \frac{5000 \times {\cdot}001}{R}\right\}}$$

$$= \sqrt{\tfrac{1}{2}\omega CR\left\{\sqrt{\left(1 + \frac{25}{R^2}\right)\left(1 + \frac{16}{10^6}\right)} + \frac{1}{250} - \frac{5}{R}\right\}}$$

This applies to any unloaded underground circuit having the constants shown in Table II.

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Now $\frac{16}{10^6}$ is so small that it may be neglected, we then have

$$\beta = \sqrt{\tfrac{1}{2}\omega CR\left\{\sqrt{\left(1+\frac{25}{R^2}\right)}+\frac{1}{250}-\frac{5}{R}\right\}} \quad . \quad (59)$$

If the quantity in the outside brackets is 1 the formula $\sqrt{\frac{1}{2}\omega CR}$ holds good exactly, if not the square root of the quantity in the outside brackets, which we will call x, enables us to make a calculation of the error involved in using $\sqrt{\frac{1}{2}\omega CR}$, thus $100(1-x)$ = the percentage error.

Example.—If the attenuation constant of the standard cable is required, the error in making use of $\sqrt{\frac{1}{2}\omega CR}$ instead of the full formula (59) is

$$100\left[1-\sqrt{\sqrt{\left\{1+\frac{25}{(88)^2}\right\}}+0{\cdot}004-\frac{5}{88}}\right]=1{\cdot}61 \text{ per cent.}$$

The effect of varying other quantities may be similarly found.

165. Attenuation Constant for Underground Air-Space Paper Cables neglecting Leakance.—If in (41) G = 0 the formula becomes

$$\beta = \sqrt{\tfrac{1}{2}\omega C(\sqrt{\omega^2L^2+R^2}-\omega L)} \quad . \quad . \quad . \quad (60)$$

This is suitable in cases where leakance is negligible, as is usually the case in unloaded underground circuits.

166. Characteristic Impedance.—The full formula is by (14), Appendix II.

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad . \quad . \quad . \quad . \quad . \quad (61)$$

This applies to all electrically long telephone circuits. If, however, ωL is great in proportion to R, and ωC is great in proportion to G, as is usually the case in loaded circuits, and also in the case of aerial unloaded circuits, if the copper conductor weight is not less than 200 lbs. per mile, Z_0 may be written

$$Z_0 = \sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

167. Wave Length Constant.—From (37), Appendix I., this is

$$\alpha = \sqrt{\tfrac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}-\tfrac{1}{2}(RG-\omega^2CL)} \quad (63)$$

If leakance may be neglected, as is usually the case, in unloaded underground circuits, this becomes

$$\alpha = \sqrt{\tfrac{1}{2}\omega C(\sqrt{\omega^2L^2+R^2}+\omega L)} \quad . \quad . \quad . \quad (64)$$

If LG = CR, and also approximately if ωL is great in proportion to R, and ωC is great in proportion to G, (63) may be written by Appendix VI. (10)

$$\alpha = \omega\sqrt{CL} \quad \text{and} \quad f = \frac{\alpha}{2\pi\sqrt{CL}} \quad . \quad . \quad . \quad (65)$$

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168. Table of Telephone Circuit Constants.—Table II. shows the calculated constants of the overhead and underground circuits usually employed in the United Kingdom. The figures given for leakance cannot, however, be deduced from the circuit dimensions in the same way as the resistance, inductance, and capacity. The leakance in underground circuits is largely a property of the dielectric (see para. 149), and the figure given, viz. 1 micromho, is equivalent to about one megohm per mile. This figure assumes a high maintenance standard, which experience has, however, shown to be quite attainable. It is now usual, on new cables, to obtain not less than 10,000 megohms per wire per mile as measured by direct currents; but at the same time, the apparent insulation to alternating currents at a frequency of $2\pi f = 5000$ is that given in the table. On overhead circuits the insulation sometimes falls below one megohm per mile, but that figure can be obtained with good maintenance for a very considerable porportion of the year (see para. 191, Chap. IX.).

All the constants given are effective values, i.e. values which hold good for alternating current measurements at the frequency given in the table.

The constants given are mean figures and the calculated attenuation constants, wave length constants, and characteristic impedances are only correct when the assumed constants obtain. In practice the capacity in underground circuits frequently differs very appreciably from the figures taken in the table, and in such cases the calculated attenuation constant requires correction; this, however, may be easily effected by means of Fig. 89.

The corrections are based on the fact that the attenuation constant by formula (58) varies directly as the square root of the capacity. The curve shows that for a capacity of 0·065, which is that taken in calculating the underground attenuation constants, the coefficient is 1; for any other value of capacity shown in the horizontal line or abscissa of the curve the capacity is given, and in the vertical column at the left coefficients are given corresponding to the different capacities. The value of the attenuation constant in the table is multiplied by this coefficient to find the attenuation constant when the capacity is greater or less than 0·065 mf.

Example.—The 20 lbs. cable with a capacity of 0·065 has an attenuation constant of 0·11643 by the table, find the attenuation constant when the capacity is 0·054. Find the capacity 0·054 in the abscissa, the corresponding coefficient in the vertical column is 0·91. We then have $0{\cdot}11643 \times 0{\cdot}91 = 0{\cdot}106$, which is the required result, and is very near the attenuation constant of the standard cable as given in the table, as it should be.

169. Changes of the Constants due to Changes of Temperature.—In open lines the resistance increases in summer as compared with winter, the effects of leakance are, however, opposite in their effect on the attenuation constant and quite mask the temperature effect, so that it is usually not considered in the United Kingdom. As regards underground circuits, the variations of temperature are much less, but in well-maintained cables they are not neutralized by varying leakance effects.

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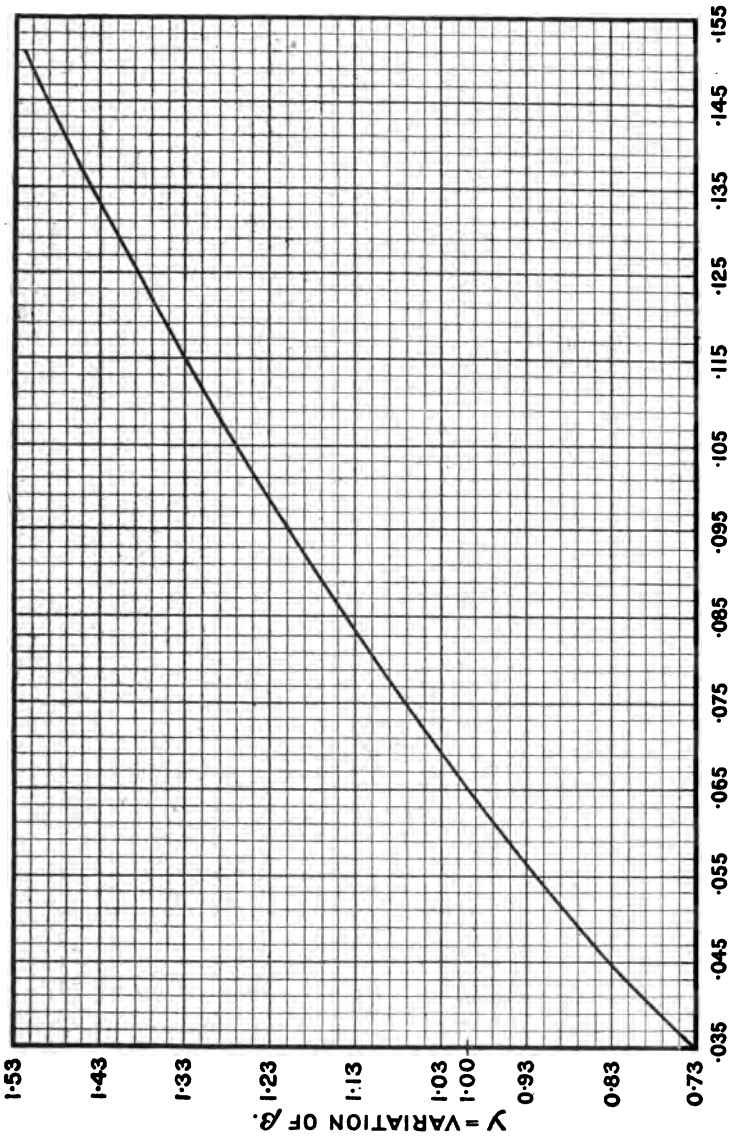

FIG. 89.—Relation between Electrostatic Capacity and the Attenuation Constant.

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TABLE I.

SIZES OF HARD DRAWN COPPER TELEPHONE CONDUCTORS (AT 60° F.). ENGLISH, AMERICAN AND CONTINENTAL.

British Standard Wire Gauge.	Brown and Sharpe Gauge. American.	Diameter. Inches.	Diameter. Millimetres.	Weight. Lbs. per mile.	Weight. Kilogrammes per kilometre.	Resistance. Ohms per mile.	Resistance. Ohms per kilometre.
4·4		0·2237	5·683	800·0	225·5	1·098	0·6825
		0·1968	5·0	620·98	175·0	1·416	0·879
5·9		0·1937	4·921	600·0	169·1	1·465	0·910
		0·1772	4·5	502·93	141·75	1·747	1·085
	6	0·162	4·1147	420·6	118·53	2·089	1·297
8·1		0·1582	4·018	400·0	112·7	2·197	1·365
		0·1575	4·0	397·4	112·0	2·212	1·374
		0·1378	3·5	304·27	85·75	2·886	1·792
9·4		0·1370	3·48	300·0	84·56	2·929	1·820
		0·1181	3·0	223·56	63·0	3·932	2·442
	8	0·1285	3·26	264·6	74·56	3·321	2·062
11·3		0·1119	2·841	200·0	56·37	4·394	2·730
		0·0984	2·5	155·24	43·75	5·663	3·517
	10	0·1019	2·59	166·4	46·91	5·28	3·279
12·6		0·0969	2·461	150·0	42·28	5·858	3·640
	11	0·09079	2·306	131·8	37·15	6·667	4·14
		0·07874	2·0	99·35	28·0	8·847	5·494
14·1		0·0791	2·009	100·0	28·19	8·787	5·460
	12	0·0808	2·05	104·6	29·48	8·4	5·22
	13	0·0720	1·829	83·09	23·42	10·58	6·57
15·7		0·0662	1·681	70·0	19·73	12·55	7·80
	14	0·06408	1·63	65·85	18·58	13·34	8·28
		0·0591	1·5	55·88	15·75	15·72	9·78
	15	0·05706	1·45	52·26	14·73	16·81	10·44
	16	0·05082	1·29	41·36	11·66	21·25	13·19
Between 17–18		0·050	1·27	40·0	11·28	21·96	13·63
	17	0·0453	1·151	32·89	9·27	26·70	16·59
	18	0·0403	1·017	26·03	7·34	33·73	20·97
		0·0394	1·0	24·84	7·00	35·37	21·97
	19	0·0359	0·91	20·30	5·79	43·26	26·57
		0·0354	0·9	20·12	5·67	43·66	27·13
Between 19–20		0·0352	0·893	20·0	5·64	43·97	27·28
		0·0315	0·8	15·90	4·48	55·2	34·33
		0·0276	0·7	12·17	3·43	72·2	44·8
	22	0·0253	0·64	10·259	2·89	85·6	53·2
Between 22–23		0·025	0·635	10·0	2·82	87·8	54·5
		0·0236	0·6	8·942	2·52	98·2	61·0
	23	0·0226	0·57	8·186	2·31	107·3	66·6
	24	0·0201	0·51	6·475	1·82	135·7	84·5
		0·0197	0·5	6·21	1·75	141·5	87·8

This table is also approximately correct for resistance according to the I.E.C. standard for annealed copper conductors at 20° C. The error is less than half of one per cent. (See para. 138.)

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TABLE II.

TELEPHONE TRANSMISSION DATA FOR CABLE AND AERIAL CONDUCTORS USED IN THE UNITED KINGDOM.

Type of line.	Constants per mile loop.				Propagation constant.	Attenuation constant.	Wave-length constant.	Characteristic impedance.	No. of miles of standard cable equal in attenuation to 1 mile of line.	No. of miles of line equal in attenuation to 1 mile of standard cable.
	R Ohms.	L Henrys.	G Micro-mhos.	C Micro-farads.						
Underground A.S.P.C. Cable.										
Standard Cable	88·00	0·001	1	0·054	0·15427 / 46° 31′	0·10616	0·11193	571·4 \ 43° 16′	1·0000	1·0000
6½-lb. cable	270·77	0·001	1	0·065	0·29668 / 45° 26′	0·20819	0·21136	912·9 \ 44° 22′	1·9611	0·5099
10-lb. ,,	176·00	0·001	1	0·065	0·23921 / 45° 43′	0·16702	0·17126	736·0 \ 44° 5′	1·5733	0·6356
20-lb. ,,	88·00	0·001	1	0·065	0·16926 / 46° 32′	0·11643	0·12284	520·8 \ 43° 17′	1·0965	0·9117
40-lb. ,,	44·00	0·001	1	0·065	0·11997 / 48° 9′	0·08004	0·08935	369·1 \ 41° 40′	0·7540	1·3263
50-lb. ,,	35·20	0·001	1	0·065	0·10749 / 48° 57′	0·07059	0·08106	330·7 \ 40° 52′	0·6650	1·5038
70-lb. ,,	25·14	0·001	1	0·065	0·09127 / 50° 32′	0·05802	0·07046	280·8 \ 39° 17′	0·5465	1·8298
100-lb. ,,	17·60	0·001	1	0·065	0·07711 / 52° 50′	0·04659	0·06145	237·3 \ 36° 59′	0·4388	2·2787
150-lb. ,,	11·73	0·001	1	0·065	0·06437 / 56° 27′	0·03558	0·05365	198·1 \ 33° 22′	0·3351	2·9840
200-lb. ,,	8·80	0·001	1	0·065	0·05735 / 59° 42′	0·02894	0·04952	176·5 \ 30° 6′	0·2726	3·6685
300-lb. ,,	5·87	0·001	1	0·065	0·05006 / 65° 7′	0·02106	0·04541	154·0 \ 24° 41′	0·1984	5·0398
Aerial Wire										
40-lb. Bronze	91·00	0·00419	1	0·0075	0·05928 / 50° 43′	0·03755	0·04857	1581·0 \ 37° 45′	0·3540	2·825
70-lb. ,,	52·00	0·00419	1	0·0079	0·04727 / 55° 15′	0·02694	0·03884	1197·0 \ 33° 18′	0·2539	3·939
100-lb. Copper	17·60	0·00390	1	0·0081	0·03259 / 68° 20′	0·01205	0·03029	804·7 \ 20° 15′	0·1136	8·803
150 lb. ,,	11·73	0·00376	1	0·0084	0·03050 / 73° 22′	0·00874	0·02922	725·3 \ 15° 16′	0·0824	12·139
200-lb. ,,	8·80	0·00366	1	0·0086	0·02955 / 76° 31′	0·00690	0·02878	687·0 \ 12° 8′	0·0650	15·375
300-lb. ,,	5·87	0·00355	1	0·0089	0·02888 / 80° 16′	0·00488	0·02846	647·3 \ 8° 26′	0·0460	21·743
400-lb. ,,	4·40	0·00344	1	0·0092	0·02858 / 82° 13′	0·00387	0·02834	620·9 \ 6° 32′	0·0365	27.412
600-lb. ,,	2·93	0·00331	1	0·0096	0·02841 / 84° 22′	0·00279	0·02827	591·6 \ 4° 25′	0·0263	38·023
800-lb. ,,	2·20	0·00322	1	0·0099	0·02836 / 85° 32′	0·00221	0·02827	572·8 \ 3° 19′	0·0208	48·008
400-lb. Iron	60·00	0·01200	1	0·0100	0·06514 / 66° 55′	0·02554	0·05993	1303·0 \ 21° 55′	0·2408	4·153

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Examples.

Example (1). Calculate the attenuation constant for a loaded 40 lbs. underground cable having the resistance and capacity shown in Table II., Chap. VIII., loaded with 200 henrys per mile $\frac{R_1}{L_1} = 50$, $\frac{G}{C} = 20$. Formula (48) may be used.

From (48)

$$\beta = \frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L}{2}\sqrt{\frac{C}{L}}$$

$$= \frac{44 + 70 \times \cdot 200}{2}\sqrt{\frac{\cdot 065 \times 10^{-6}}{\cdot 200}}$$

$$= \frac{44 + 14}{2} \times 0{\cdot}000571$$

$$= 29 \times 0{\cdot}000571 = \underline{0{\cdot}01653} \text{ hyps. per mile loop.}$$

Example (2). Calculate the modulus of the characteristic impedance for the same constants.

In this case a sufficiently near approximation to the correct result is obtained by the use of formula

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0{\cdot}200}{0{\cdot}065 \times 10^{-6}}} = 10^3\sqrt{\frac{200}{65}} = \underline{1754} \text{ ohms}$$

Example (3). Find the percentage increase in the resistance of a hard-drawn 600 lbs. copper conductor at a frequency of 800 cycles per second as compared with the resistance when measured by direct current.

By formula (25)

$$Z = 2\pi^2\mu\kappa r^2 f$$

Where μ for copper is unity, κ is the conductivity in C.G.S. units per cubic cm. $= \frac{1}{1731}$, r is the radius of the wire in cm. $= \frac{0{\cdot}4921}{2}$ from Table I., $f = 800$.

$$\therefore Z = \frac{9{\cdot}87 \times 2 \times (\cdot 246)^2 \times 800}{1731}$$

$$= \frac{19{\cdot}74 \times 48{\cdot}4}{1731} = \frac{955}{1731} = 0{\cdot}55$$

From Fig. 2, when $Z = 0{\cdot}55$ $\frac{R_1}{R} = 1{\cdot}025$ approximately.

The required percentage increase is therefore $2\frac{1}{2}$ per cent.

Example (4). Find the self-inductance in henrys of a mile of double-wire circuit, measured by direct current, of a 400 lbs. commercial iron wire when the wires are 17 inches apart on the poles, and the permeability of iron is taken at 100. The diameter of the wire may be taken as 171 miles.

Formula (30) applies in this case. We have

$$L = 4l\left(\log \frac{D}{r} + \frac{\mu}{4}\right)$$

if $\frac{D}{l}$ is relatively small, as is the case.

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The portion $4l \log \frac{D}{r}$ may be written as in formula (31), *i.e.* $0{\cdot}0014822 \log_{10} \frac{D}{r}$. The second portion is $4l\frac{\mu}{4}$, where l is expressed in cm. Converting to express L in henrys per mile we have $l\mu = 161000 \times 100 = 16100000$, and dividing by 10^9 to reduce to henrys we have $l\mu = 0{\cdot}0161$.

The value of L is therefore

$$L = {\cdot}0014822 \log_{10} \frac{D}{r} + {\cdot}0161$$

that is

$$0{\cdot}0014822 \log_{10} \frac{17{\cdot}0}{\frac{{\cdot}17}{2}} + 0{\cdot}0161 = {\cdot}0014822 \log_{10} \frac{3400}{17} + 0{\cdot}0161$$

$$= 0{\cdot}0014822 \times 2{\cdot}30103 + 0{\cdot}0161 = 0{\cdot}0034 + 0{\cdot}0161$$

$$= \underline{0{\cdot}0195 \text{ henry}}$$

Example (5). If a frequency of 800 cycles per second is applied to this circuit, find (*a*) the effective resistance per mile; (*b*) the effective inductance per mile. Having given that the resistance of commercial iron is 12140 C.G.S. units per cub. cm.

(*a*) From formula (25) we have

$$Z = 2\pi^2 \mu \kappa r^2 f$$

as in the previous example.

Whence $Z = 19{\cdot}74 \times 100 \times \frac{1}{12140} \times \left(\frac{{\cdot}170 \times 2{\cdot}54}{2}\right)^2 \times 800$

$$= \frac{1974 \times 0{\cdot}0467 \times 800}{12140} = \frac{73955}{12140} = 6{\cdot}06$$

This is the value of Z, and from Fig. 87 the corresponding value of $\frac{R_1}{R} = 2{\cdot}03$.

By formula (23) the resistance of 400 lbs. iron is

$$\frac{5328}{400} \times 2 = 13{\cdot}3 \times 2 = 26{\cdot}6$$

The effective resistance at 800 frequency is therefore

$$\frac{R_1}{R} = \frac{R_1}{26{\cdot}6} = 2{\cdot}03$$

Whence $\underline{R_1 = 54 \text{ ohms per mile.}}$

(*b*) The ratio $\frac{L_1}{L}$ by Fig. 88, with a value of $Z = 6{\cdot}06$ is $0{\cdot}564$. Now, by example (4) the inductance when calculated for direct current is $0{\cdot}0034 + 0{\cdot}0161$, which now becomes

$$0{\cdot}0034 + 0{\cdot}0161 \times 0{\cdot}564 = \underline{0{\cdot}012 \text{ henry per mile.}}$$

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CHAPTER IX

THE LOADING OF TELEPHONE CIRCUITS—IDEAL, CONTINUOUS, SERIES COIL LOADING AND LEAK COIL LOADING—A.C. CASE

170. Effects of Electrostatic Capacity in Transmission Lines.—Modern commercial loading is at present far from attaining the full theoretical benefits of ideal loading. The commercial form will be more easily understood after obtaining a clear idea of the ideal type, and this will now be briefly explained. Consider in the first instance the transmission of direct currents in a transmission line having uniformly distributed resistance and leakance. The characteristic impedance and attenuation constant in such a case are given by formulæ (5) and (6), Chap. II. Concrete examples of the magnitude and effects of the impedance and attenuation constant are given by Table I., Chap. II. Next suppose that the circuits dealt with in this table are underground circuits having the constants shown in Table II., Chap. VIII. The latter table shows the effects of sine wave transmission at 800 cycles per second. In these circuits, therefore, if a direct voltage is applied to the circuit, $R_0=\sqrt{\frac{R}{G}}$ and $\beta=\sqrt{RG}$ whereas if an alternating voltage of sine wave form at 800 cycles per second is applied to the same circuit we have $Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$ and

$$\beta=\sqrt{\tfrac{1}{2}\{\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+GR-\omega^2CL\}}$$

by (15) and (36), App. I., and these are the formulæ applicable to Table II., Chap. VIII. If L and C are each zero in the A.C. case, we then have $Z_0=\sqrt{\frac{R}{G}}$ and also $\beta=\sqrt{RG}$, as they do in the direct current case for the same values of R and G. It therefore appears that the complexity of the A.C. formulæ is due to the circuits having inductance and capacity. So far as the relative effects of L and C are concerned in the examples taken, the capacity has by far the greater effect on the result, particularly in the case of the circuits having the higher resistance; in fact, the elimination of L in the case of the 40 lbs. circuit only affects the results in a relatively small degree. It follows, therefore, that the more unfavourable attenuation constant is practically due to the effect of capacity, and this is to a great extent true of the 70 lbs. and 150 lbs. conductors. The result

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is seen from a comparison of the characteristic impedance and attenuation constant in the D.C. and A.C. cases in Table I., Chap. II., and Table II., Chap. VIII. In each instance the impedance is much less in the A.C. case and the attenuation constant much greater in that case. The effect of the capacity is to cause the volts to be out of phase with the current at every point in any infinite line, the negative angle being that of the characteristic impedance. For example, in the case of the 40 lbs. circuit, by Table II., Chap. VIII., the angle is $\angle 41° 40'$; this results in a small power factor; energy is also lost in charging the dielectric. The resultant effect on the attenuation constant and on Z_0 varies with frequency, and distortion of speech therefore also results.

171. Compensation of the Effects of Capacity by Inductance—The Distortionless Circuit.—It is theoretically possible to completely eliminate the effects of capacity in a transmission line by the addition of uniformly distributed inductance (O. Heaviside, "Electromagnetic Theory," 1894, vol. i. chap. iv.). This result is brought about when

$$LG = RC \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In that case $\beta = \sqrt{RG}$ and $Z_0 = \sqrt{\frac{R}{G}}$, and the attenuation constant has then the minimum value for a given value of R and G (see App. V.). When these relations are established the attenuation constant and characteristic impedance are evidently, by what has been shown in the preceding paragraphs, exactly the same as in the DC case *for the same constant values of* R *and* G. The volts and current in the A.C. case are now everywhere in phase, in an infinite line, and the attenuation constant is the same for all frequencies. This is indicated by the fact that the frequency factor ω has disappeared from the formulæ for Z_0 and β. The losses in such a case are unavoidable energy losses (*see* formula (24) and para. 149, Chap. VIII.), and cannot be reduced unless R and G are reduced in magnitude.

It may assist to understand the effect of inductance, if it is remembered that if the circuit had no capacity, but that the same amount of inductance as is necessary for correcting the capacity were uniformly distributed throughout the circuit, the impedance would be *increased* as compared with a *reduction* when capacity alone is added, and Z_0 would everywhere have a positive angle equal in magnitude but opposite to that caused by capacity in the absence of the inductance. The proper amount of inductance, therefore, corrects the magnitude of the characteristic impedance and its angle.

Now although a given circuit cannot have a less attenuation constant for alternating current transmission than that obtained in direct current transmission, with a constant value of R and G, it does not necessarily follow that this is the most favourable value of β for such a circuit; this will evidently depend on the leakance, and this is exemplified by the remarks after (42), Chap. VIII. The function of ideal loading is to correct the capacity effect and give the minimum β for given values of R and G. The effect in series loading is always to increase the impedance and to reduce β. It follows from this that unless a given circuit is long enough, there will be no benefit

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from loading, because the effect of the increased impedance on the sent current may more than counteract the beneficial effect of the smaller β.

172. Effect of Leakance in Loaded and Unloaded Circuits.—In an unloaded circuit the characteristic impedance is $Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$, as already noted, whereas in the ideal loaded circuit we have $Z_0 = \sqrt{\frac{R}{G}}$. Now it may be seen by inspection of Table II., Chap. VIII., that for the values of G and C taken for the calculation of underground circuits ωC is 325 times greater than G, and it will be seen by inspection of the formula for Z_0 in the unloaded case that the leakance has very little effect on the result, for moderate variations of G. If, however, the leakance varies when the circuit is fully loaded in the ideal case, Z_0 is inversely proportional to the square root of G. The circuit impedance is therefore more sensitive to leakance changes when fully loaded. Now the attenuation constant, by (52), Chap. VIII., may be expressed in terms of the characteristic impedance in the case of loaded circuits, and from what has been said it will be largely affected by leakance; whereas in the case of unloaded circuits an examination of para. 164, Chap. VIII., will show that the attenuation constant in the unloaded case is very little affected by leakance. It may be remarked that in all types of loaded circuits the attenuation constant is more affected by leakance than is the case with unloaded circuits. This may be seen by (44), Chap. VIII., which shows that the second term of β is proportional to $G\sqrt{L}$, *i.e.* it increases with G and with $\sqrt{L}$.

173. Analogy between Ideal Loading and Pure Sine Wave Transmission.—When $LG = CR$ we obtain, by rearranging the equation and taking the square root of each side

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Each side has the dimensions of an impedance, and it is easily shown that $\sqrt{\frac{R}{G}}$ is the characteristic impedance, for we have by substitution of the values of L and C in the full formula from (1)

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{R + j\frac{\omega RC}{G}}{G + j\frac{\omega LG}{R}}} = \sqrt{\frac{RG + j\omega RC}{RG + j\omega LG}}\sqrt{\frac{R}{G}}$$

$$= \sqrt{\frac{R}{G}} \quad \text{since } RC = LG \text{ by (1).}$$

Now $$Z_0 = \frac{E_1}{I_1} = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}}$$

(2) may therefore be written

$$E_1^2 G = I_1^2 R$$

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But I_1^2R is the energy loss in the conductor and $E_1^2G = \frac{E_1^2}{R_2}$ $= E_1I_2$ is the energy loss in the dielectric if $G = \frac{1}{R_2}$ and $\frac{E_1}{R_2} = I_2$. So that for the distortionless circuit (ideal loading) we have equal consumption of energy in the conductor and the dielectric.

If (2) be written in the form

$$\frac{E_1}{I_1} = \sqrt{\frac{L}{C}}$$

we have $$E_1^2C = I_1^2L \quad \text{or} \quad \tfrac{1}{2}CE_1^2 = \tfrac{1}{2}LI^2 \quad . \quad . \quad . \quad (3)$$

This equation shows the relation of the magnetic energy $\frac{1}{2}LI^2$ to the electric energy per unit length in an electromagnetic wave: it is distortionless, and if the waves are supposed in this case to be subject neither to resistance nor leakance, they are not subject to attenuation. This is an example of Poynting's Law that the flow of energy in an electromagnetic field is at right angles at every point both to the magnetic and electric components of the field and is proportional to the product of the two field intensities (J. Poynting, *Phil. Trans.*, 1884, Part II., p. 343, etc.). When the relation in (3) obtains we have pure sine wave transmission, and the telephone circuit with ideal loading gives the nearest approach to it with wires of resistance R and leakage G.

174. Practical Alternatives to Ideal Loading.—In practice ideal loading is never attained. Ideal loading supposes the added inductance to have no resistance, but this is not practicable and the added resistance materially affects the result. Again, in the preceding considerations we have supposed the leakance of the circuit to remain constant at all frequencies, but owing to the reasons given under the heading of "Leakance," para. 149, this is far from being the fact. If we suppose the formula for the ideal β to be retained, then both R and G must be supposed to be materially increased, and further both R and G will vary with frequency, so that the circuit cannot be distortionless as supposed in ideal loading.

This brings us to a further consideration. It would frequently be impossible in practical loading on any system to provide and maintain the large amount of inductance required to produce the ideal result. Thus in the case of the standard cable more than one henry per mile would be required. It will further be shown that if this large amount of inductance could be satisfactorily inserted and maintained, it would be at present far from economical to provide it (see Chap. XII.). We have, however, fortunately a practical alternative to ideal loading, which is based on the fact that if ωL is great in proportion to R and ωC is great in proportion to G the fundamental advantages of ideal loading can be obtained, even when the amount of inductance employed is very far from that required for the ideal condition (*see* Appendix VI. for proof of the relative formula).

It should be noticed that the impedance formula is the same whether we have pure sine wave transmission, ideal loading, or ordinary commercial loading; this is seen by comparing (2) and (3) in this chapter with (62), Chap. VIII. We have in each case what

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amounts to $\frac{E}{I}=\sqrt{\frac{L}{C}}$. In commercial loading an approximation to the magnitude of $\sqrt{\frac{L}{C}}$ is obtained even when the amount of inductance is one-tenth or more of the amount required for the minimum attenuation constant, but the angle between the volts and current gradually increases as the conditions for the minimum attenuation constant are departed from. In practice the wide range of inductance which permits of greater or less approximation to the best attainable transmission result, permits of choice in the degree of improvement of transmission according to the needs of the telephone service, and in the cost of adding the inductance to provide the improvement (see Chap. XII.).

The variation of resistance with the addition of inductance and the variation of leakance with frequency, in addition to the variation in the quantity of inductance, which has been shown to be permissible, result in the formulæ for ideal loading being modified, and the changes are developed in formulæ (42) to (56), in Chap. VIII.

Loading also results in the speed of propagation of the waves in the circuit being slowed down, and the changes in the formulæ are developed in App. VI.

175. Methods of adding Artificial Inductance to Transmission Lines.—The following three methods will be considered in detail:—

(*a*) **Continuous Loading.**—In this type of loading the inductance is provided by one or more windings of magnetic material such as iron or stalloy, wound continuously round the conductors.

(*b*) **Lumped Series Loading.**—This has already been dealt with, for the direct current case, in Chap. IV.

(*c*) **Lumped Leak Loading.**—This has also been dealt with, for the direct current case, in Chap. IV.

Both (*b*) and (*c*) will in turn be adapted to the alternating current case.

Before considering these various types of loading in detail, an important general relation between β_m and the amount of inductance used will be developed.

176. Relation between the Minimum Attenuation Constant and the Degree of Loading.—By (50), Chap. VIII., we have $R=\left(\frac{R_1}{L_1}+\frac{G}{C}\right)L$

or
$$L=\frac{R}{\left(\frac{R_1}{L_1}+\frac{G}{C}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Here L is the amount of inductance per mile required to produce the mimimum attenuation in a conductor of R ohms per mile loop. $\frac{R_1}{L_1}$ is the ratio of the effective resistance R_1 added to the circuit by the inductance L_1 when $2\pi f=5000$; and $\frac{G}{C}$ is the ratio of the leakance G to the capacity C at the same frequency (para. 149, see Chap. VIII.). It is assumed that $\frac{R_1}{L_1}$ and $\frac{G}{C}$ are constants, *i.e.* that R_1 varies as L_1 varies and G varies as C varies over a range of values

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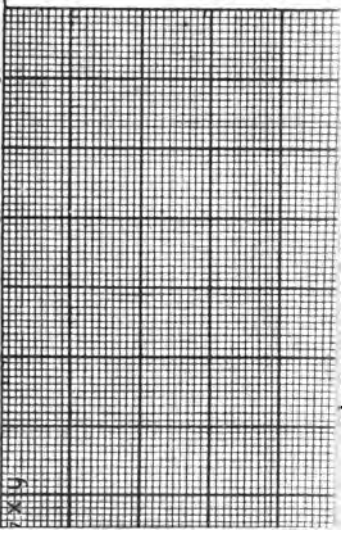
WEIGHT OF CONDUCTOR PER MILE PER WIRE IN LBS.				y
20	40	50	70	
0·0380	0·0268	0·0241	0·0203	1·9
0·0370	0·0262	0·0235	0·0198	1·85
0·0360	0·0255	0·0228	0·0193	1·8
0·0350	0·0248	0·0222	0·0187	1·75
0·0340	0·0241	0·0215	0·0182	1·7
0·0330	0·0233	0·0209	0·0176	1·65
0·0320	0·0226	0·0203	0·0171	1·6
0·0310	0·0219	0·0196	0·0166	1·55

$\beta m = \sqrt{CR\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} =$ ✓



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met with in practice, and it is known that this is fairly correct for $\frac{G}{C}$ and for $\frac{R_1}{L_1}$ within the limits here applied.

If (48), Chap. VIII., be divided by (51), Chap. VIII., we have

$$\frac{\beta}{\beta_m}=\frac{R+\left(\frac{R_1}{L_1}+\frac{G}{C}\right)L_2}{2\sqrt{CR\left(\frac{R_1}{L_1}+\frac{G}{C}\right)}}\sqrt{\frac{C}{L_2}}=\frac{\sqrt{\frac{L}{L_2}}+\sqrt{\frac{L_2}{L}}}{2}=y \quad (5)$$

Here L is the inductance required to produce β_m, and L_2 is any other inductance, limited in amount only by the fact that it must be sufficiently large for ωL to be so great in proportion to R that formula (48), Chap. VIII., holds good. For proof see App. VII.

From (5) we have

$$\beta=\beta_m\left(\frac{\sqrt{\frac{L}{L_2}}+\sqrt{\frac{L_2}{L}}}{2}\right)=\beta_m y \quad . \quad . \quad . \quad (6)$$

It is evident from (6) that the relation between β_m and β for any circuit does not depend on the absolute values of L and L_2, but only on their ratio, and for the same fraction $\frac{L}{L_2}$ the value of β is always a constant quantity greater than β_m. For any given conductor, however, the value of L can be found by means of (4) if $\left(\frac{R_1}{L_1}+\frac{G}{C}\right)$ is known. For a fixed value of $\left(\frac{R_1}{L_1}+\frac{G}{C}\right)$ the magnitude of L is directly proportional to R by (4). If a curve is made by plotting $\frac{L}{L_2}$ as abscissæ and $\frac{\sqrt{\frac{L}{L_2}}+\sqrt{\frac{L_2}{L}}}{2}$ as ordinates the curve will give multipliers in the ordinate that will enable us to find β for any circuit having a minimum attenuation constant β_m and a degree of loading $\frac{L}{L_2}$.

The principle outlined in the preceding paragraph is applied in Fig. 90. Here $\frac{L}{L_2}$ is given in the horizontal line and

$$\frac{\sqrt{\frac{L}{L_2}}+\sqrt{\frac{L_2}{L}}}{2}=y$$

in the vertical column, and the graph connects the two sets of values. In the graph L is called L_{max} and L_2 is called x. Immediately under the values of $\frac{L}{L_2}$ seven rows of horizontal figures are given. These are the amounts of inductance corresponding to $\frac{L}{L_2}$ for the conductor shown at the left-hand end of the line in each case. Thus for a 100-lbs. conductor when $\frac{L}{L_2}=1$ we have the inductance required

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for β_m for that conductor, *i.e.* 250 millihenrys. When $\frac{L}{L_2} = 5$ we have only one-fifth of this amount, or 50 mh., and accordingly under $\frac{L}{L_2} = \frac{L_{max}}{x} = 5$, the value 50 mh. will be found associated with the value $\frac{L_{max}}{x}$; when $\frac{L_{max}}{x} = 10$ we have $\frac{250}{10} = 25$, and so on. The same arrangement applies to all the conductors shown. As regards the vertical columns where the same conductors are arranged in order, when $\frac{L_{max}}{x} = 1$, we have β_m, and accordingly the values of β_m are shown in the vertical columns of all the conductors opposite $y = 1$. When $y = 1{\cdot}5$ the attenuation constants of all the conductors are $\beta_m \times 1{\cdot}5$, which by (6) gives β. In this way all possible values of β for all the conductors in the graph are given over a range of inductance varying from $\frac{L}{L_2} = 1$, to $\frac{L}{L_2} = 12$. The table assumes that $C = 0{\cdot}065$ mf. and $\left(\frac{R_1}{L_1} + \frac{G}{C}\right) = 70$. If it does not, correcting graphs with a full explanation of their use, and also examples of the method of using Fig. 90, are given in App. VII. The following is, however, an example of the use of Fig. 90. A 70-lbs. conductor is loaded with 55 mh. per mile. Its capacity is 0·065 mf. and $\left(\frac{R_1}{L_1} + \frac{G}{C}\right) = 70$. What is the attenuation constant? At the foot of the table find 70 lbs. in the extreme left-hand column and run along the horizontal line until 55 is reached. If a vertical line is traced from 55 to the graph it will be found to be opposite 0·0157 in the vertical column appropriated to the 70 lbs. conductor, and 0·0157 is the attenuation constant required.

177. Continuous or Krarup Loading.—This type of loading was first proposed and introduced into working circuits by C. E. Krarup, Engineer-in-Chief of the Danish Telegraph Administration (*E.T.Z.*, p. 344, 1902). Formulæ permitting of the calculation of the inductance and effective resistance added to telephone conductors by the wire or tape which is wound spirally round the conductor to produce this type of loading, have been developed by Prof. Larsen (*Elektrotechnische Zeitschrift*, 1908, p. 1030).

The inductance of a continuously loaded cable is given by the following formula, which is developed and proved in Prof. Breisig's "Theoretische Telegraphie," p. 322 :—

$$L = 4l\left\{\log\frac{2d_1}{r} + \frac{1}{4} + \mu\frac{\pi t^2}{2(2r+t)(t+a)(1+\alpha)}\right\} \quad . \quad (7)$$

where l is the length in cm., $2d_1$ is the distance between the centres of the conductors, r is the radius of the conductor, μ is the permeability of the iron, t is the thickness of the iron winding, a is the distance between the wire spirals, and α is a factor which accounts for energy losses due to the resistance of the magnetic circuit.

In continuous loading an increase of magnetic flux is obtained owing to the fact that the circular lines of force round the wire

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find a better conductor through the iron winding than through air. At the same time, however, owing to the spiral form of the winding, the circular lines of force meet air gaps in the iron and these increase the resistance of the magnetic path. The resulting energy losses, including hysteresis and eddy current losses, can only be expressed approximately, and they and the permeability are best determined by experiment, and previous knowledge of the type of material used. When, however, correct data are obtained the formulæ here given enable us to predict the result of varying the dimensions of the wire, and its permeability.

If the losses due to a and α are neglected, formula (7) becomes

$$L = 4l\left\{\log \frac{2d_1}{r} + \frac{1}{4} + \frac{\mu\pi t^2}{2(2r+t)t}\right\} \quad . \quad . \quad . \quad (8)$$

and this formula was found in practice to hold approximately good by M. Devaux Charbonnel, in a series of experiments made by him (see "La Lumière Eléctrique," 15th June, 1912).

In order that (8) may be sufficiently accurate it is necessary that a and α should be small in comparison with $2(2r+t)t$. It is evident that if the convolutions of wire are wound closely together a will be very small and that α will be reduced, as the resistance of the magnetic circuit will be decreased.

The first part of (8), viz. $4l\left(\log \frac{2d_1}{r} + \frac{1}{4}\right)$ accounts for the natural inductance of the circuit apart from the iron, and the added inductance is therefore—

$$L = 4l\left\{\mu \frac{\pi t}{2(2r+t)}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Since $2r = d$ (the diameter of the wire), we get

$$L = \frac{2\mu\pi lt}{(d+t)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If the thickness of iron be made up by p windings, each of thickness t, this becomes $L = \frac{2\pi\mu lpt}{(d+pt)}$, but

$$\frac{pt}{(d+pt)} = \frac{1}{\frac{d}{pt}+1} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and from this it may be seen that if the ratio $\frac{d}{pt}$ is kept constant, L will be unaltered (if the natural inductance of the circuit is unaltered). On that assumption it is evident that, in order to obtain the same value of inductance in conductors of different diameters, the thickness of the iron winding (pt) must be increased in the same proportion as the diameter of the copper conductor is increased. The expression for L is in absolute units, and to bring it to henrys per km. we have

$$L = 10^{-4} \times 2\pi\mu \times \frac{pt}{(d+pt)} \quad . \quad . \quad . \quad (12)$$

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The figure to be taken for permeability, μ, is a somewhat doubtful quantity. In practice it is found to vary between 100 and 200 or more at present.

The increase in resistance brought about by the insertion of inductance in continuously-loaded cables has been made the subject of exhaustive research by Prof. Larsen; and Prof. Breisig, on p. 323 of "Theoretische Telegraphie," gives the following formula after consideration of Larsen's work :—

$$R_2 = \frac{S_e \pi^2 \mu^2 \omega^2 t^4}{4(2r+t)(t+a)(1+a)^2} \quad . \quad . \quad . \quad (13)$$

where S_e is the specific conductivity in C.G.S. units of the iron, t is the thickness of the iron winding, r is the radius of the wire on which it is wound, $\omega = 2\pi f$, μ is the permeability, and a and a are correcting factors as in (7). Neglecting these losses, we get

$$R_2 = \frac{S_e \pi^2 \mu^2 \omega^2 t^3}{4(2r+t)} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Expressing the specific conductivity S_e in terms of specific resistance and rearranging the formula, we get

$$R_2 = \frac{4\pi^4 \mu^2 f^2 t^3}{4\rho(d+t)} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where d = the diameter of the conductor, and all the dimensions are in centimetres. To convert to ohms per kilometre we have

$$R_2 = 10^{-4} \frac{\pi^4 \mu^2 f^2 t^2}{\rho} \cdot \frac{t}{d+t} \quad . \quad . \quad . \quad . \quad (16)$$

It is evident by (10) that the effect of a thin winding t is relatively greater than a thick one to a certain extent. This is evident from the fact that the inductance added by the iron is proportional to $\frac{t}{(2r+t)}$ or $\frac{t}{(d+t)}$. If t is very thin the inductance added will be nearly proportional to $\frac{t}{d}$, whereas if t is of appreciable thickness by comparison with d the inductance is proportional to $\frac{t}{(d+t)}$.

Thus the inductance due to a wire 1 mil thick added to a conductor having a diameter of 100 mils is proportional to $\frac{t}{d+t} = \frac{1}{101} = \frac{0{\cdot}99}{100}$

If the thickness of iron wire were 20 mils we should have

$$\frac{t}{d+t} = \frac{20}{120} = \frac{16{\cdot}66}{100}$$

At the same rate 1 mil would give $\frac{16{\cdot}66}{100} \times \frac{1}{20} = \frac{0{\cdot}833}{100}$ instead of $\frac{0{\cdot}99}{100}$ as in the previous case.

An inspection of (16) shows that the effect of the thickness t on the resistance depends to some extent on the diameter of the

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conductor on which it is wound, as R_2 is inversely proportional to $(d+t)$; if the thickness t is very small in comparison to d the resistance varies nearly as the third power of t, which shows that fine wire should very materially be less in resistance than thick wire. The usual range of thickness is at present from 8 to 12 mils, but very thin and relatively wide tapes have been used; the effective resistance due to the iron was relatively high in this case, but the increased permeability obtained resulted in the production of a low attenuation constant. Formula (16) also shows that a material having high specific resistance is advantageous if it can be combined with high inductance, in view of the fact that the resistance R_2 varies inversely as the specific resistance. R_2 in (16) varies, however, as the square of the permeability.

178. Effect of an increase in R_2 and L on the Attenuation Constant.—In considering the effect of the changes of R_2 and L it is necessary to look beyond the production of inductance to its main object, which is the reduction of the attenuation constant, regard being had at the same time to the suppression of undue distortion.

Consider the effect on a given conductor, of increasing the inductance by 50 per cent. and at the same time of doubling the effective resistance added by the iron. Suppose we have a conductor with the following constants:—

R per mile loop = 25 ohms.
L ,, ,, = 12 mh.
C ,, ,, = 0·060 mf.

Suppose that when $2\pi f = 5000$ the ratio of inductance added to resistance is $\frac{R_1}{L_1} = 60$ and that $\frac{G}{C} = 20$. Then by formula (48) Chap. VIII., we have

$$\beta = \frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L}{2}\sqrt{\frac{C}{L}} = 10^{-3}\left(\frac{25 + (60 + 20)0{\cdot}012}{2}\right)\sqrt{\frac{0{\cdot}06}{0{\cdot}012}}$$

$$= \frac{25{\cdot}96}{2}\sqrt{5} \times 10^{-3} = \underline{0{\cdot}029}$$

Suppose now that some different known material has a much higher ratio $\frac{R_1}{L_1}$—assume that it is doubled—whilst at the same time the inductance, with the same thickness of iron, is increased by 50 per cent. In that case we have $\frac{R_1}{L_1} = 120$ and $L = 0{\cdot}018$, the other constants being as before. It will be found that the attenuation constant is now 0·0257 instead of 0·029 as in the former case. It will be seen, therefore, that although the resistance is increased much more than the inductance the net result is a diminution of β. It is therefore obviously desirable to take calculations of β directly into account in determining the dimensions and properties of the iron wire.

179. Advantages and disadvantages of Continuous Loading. Advantages.—(*a*) The principal advantage of continuous loading is

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that any length of such a cable, no matter how small, constitutes a self-contained unit of loaded cable, without the necessity of inserting coils as in lump-loading. This is an advantage for submarine cables, and consequently we find a relatively large number of continuously loaded cables utilized for this purpose.

(*b*) The fact that any small length is uniformly loaded also renders this type of cable suitable for insertion as short lengths of underground line which occur in aerial lines, where owing to local circumstances aerial line is impracticable. The continuously loaded cable in such a case forms an alternative to the use of unloaded underground cable. Lump-loading does not attain its maximum efficiency in such a case.

(*c*) The impedance of a long length of continuously loaded cable should vary less than a coil loaded cable, firstly because it is uniformly loaded, and secondly because it is not subject to the variations of impedance which occur in coil-loaded cables due to the fact that it is not always practicable to place the first loading coil at the most suitable point, and the impedance of such a cable varies largely with the position of the coil.

The uniform impedance of telephone conductors is of special importance in phantom circuit working, and in the working of telephone relays. In a four-wire submarine cable if all the wires are well balanced throughout it is possible to obtain two direct or side circuits and two phantom circuits as shown later. The second or earthed phantom circuit has not up to the present been satisfactorily worked in a coil loaded cable, but it has in a continuously loaded cable.

Disadvantages.—(*a*) The amount of inductance which can be produced by this method is at present limited; for example, 20 millihenrys per mile per pair is a good result. There is also a possibility of unequal *ageing* in the iron winding of the two wires of a circuit.

(*b*) The effective gauge of the conductor is increased by the iron wire, and this increases the dimensions and cost of the cable both for underground and submarine cables. It is particularly expensive in the case of submarine cables, because for the same mechanical strength, the thickness of guttapercha must be materially increased, as compared with the amount required for unloaded conductors. This point will be dealt with later.

(*c*) This method is limited to new cables in its application, whereas existing ones frequently require to be loaded.

(*d*) The electrostatic capacity of a loaded wire is somewhat increased as compared with the same unloaded wire, but the increase is not usually serious.

180. Distortion in Continuously Loaded Cables.—In view of the fact that the effective resistance of cables of this type varies both with frequency and current, and that the leakance varies with frequency, it is evident that such cables are not distortionless. They have usually a higher value of $\frac{R_1}{L_1}$ than loaded cables, but do not of course suffer from the disadvantage due to lumped inductance. For equal values of $\frac{R_1}{L_1}$ their distortion is less than that of coil-loaded

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cables. The fact that this type of cable has a relatively low value of inductance is favourable from the point of view of distortion, as the distortion is likely to be greater for heavily loaded than for lightly loaded cables. This is clear from para. 172. Some details of the relative distortion in continuously loaded and coil loaded submarine cables are given by Dr. Petritsch in an article on the Adriatic continuously loaded cable in *Electrotechnik und Maschinenbau*, 6th July, 1913, p. 595. It was found that the continuously loaded cable had a distortion 11 per cent. less than the Anglo-Belgian coil loaded cable between the limits $2\pi f = 3000$ to $2\pi f = 7000$. It is, however, necessary to take into account the relative values of $\frac{R_1}{L_1}$ and the variation of R with frequency in every case and compare the increase of attenuation with frequency. It is not advisable to generalise too much.

In view of the fact that the energy losses due to hysteresis increase directly as the frequency and those due to eddy currents vary as the square of the frequency, the rate at which the effective resistance increases with frequency must depend on the relative magnitudes of the two losses. In some cases the observed results point to an increase of resistance in the neighbourhood of the square of the frequency in continuously loaded cables.

181. Series Lumped Loading. Alternating Current Case.—The formulæ for the characteristic impedance and attenuation constant of loaded circuits (61) and (48), Chap. VIII., have been developed on the assumption that the electrical constants of the loaded circuit, including the inductance, are uniformly distributed. It was, however, shown by Prof. M. I. Pupin that if inductance is inserted in lumped quantities in series with the conductor at regular distances, according to a law given by him in the paper named, the benefits of distributed loading can be attained. The law referred to is given in this chapter in para. 182. The full theory is developed by Pupin in a paper given before the American Institution of Electrical Engineers on March 22nd, 1899, entitled "Propagation of Long Electrical Waves." See also British Patent Specification 12733, October, 1900. Dr. G. A. Campbell, of the Engineering Department of the American Bell Telephone Co., also independently applied for a patent in America in March, 1900, but priority was given to Dr. Pupin. Dr. Campbell also wrote an article "On Loaded Lines in Telephonic Transmission," in the *Philosophical Magazine*, vol. v. p. 313 (1903), which includes the formulæ often associated with his name. Oliver Heaviside appears, however, to have been the first to deal with the theory of distributed inductance (vol. xix. p. 79, *Electrician* (1887)). Vaschy in France independently investigated the subject and published details of his work shortly afterwards. Series lump loading may be treated as an application of the equivalent circuit as developed in Chap. IV., the only difference in treatment is that all the quantities in the alternating current case are vectors. This method of development will now be applied to practical series loading, and a few of the more important facts will be briefly recalled at this stage. It is proved in Apps. III. and IV. that every uniform transmission line of any length may be represented for electrical operations conducted from its two ends

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at a fixed frequency (for measurements of impedance and tota[1] attenuation between its two ends), by an equivalent π or T circuit, and conversely it follows that any given π or T circuit has some corresponding uniform line to which it is equal in the conditions just defined; examples of such pairs of circuits are given in Chap. IV. If any number of exactly similar equivalent circuits are connected in series, the impedance and total attenuation from the two ends of the circuit thus formed are the same as from the ends of the same number of corresponding lengths of uniform line placed in series, for one frequency.

In practical commercial loading, loading coils are inserted at equal distances in a uniform transmission line, and the following is a practical case. A 70 lbs. pair of underground conductors having the constants given in Table II., Chap. VIII., has loading coils inserted at intervals of $2\frac{1}{2}$ miles, except at the ends, where the cable length is $1\frac{1}{4}$ miles in both instances. Such a cable is uniformly loaded. This will be seen by noticing that each loading coil may be regarded as loading half of the $2\frac{1}{2}$ mile cable section spaced equally on both sides of a loading point, or in other words, half the inductance in the loading coil loads $1\frac{1}{4}$ miles of line. Viewed in this way it will be seen that every $1\frac{1}{4}$ mile of cable throughout the line is associated with half a loading coil, including the two end half sections of $1\frac{1}{4}$ miles. If two such sections be combined, either by placing a coil in the middle of a $2\frac{1}{2}$ mile section to represent the two loaded half sections, or joining two $1\frac{1}{4}$ mile lengths of cable together to form a $2\frac{1}{2}$ mile section and adding half a coil at each end (as shown in Fig. 111), we obtain a complete loading coil section. It will be convenient to investigate this latter case on the assumption that a complete coil has an inductance of 132 mh. and each half coil will be assumed to have half of this inductance. Moreover, as we are dealing with a double wire circuit, the inductance will be divided between the A and B lines. The effect on transmission of the insertion of the loads is worked out as an equivalent circuit problem at the end of the chapter, and it will be seen that the length of 70 lbs. loaded cable giving the same total attenuation as the unloaded 70 lbs. cable in Table II., Chap. VIII., is 3·62 times as great at a frequency such that $2\pi f = 5000$, and it may be noted that speech tests give a similar result. Now although the equivalent circuit is equal to the real line in the conditions stated, at a frequency of 800 cycles per second, it does not by any means follow that it will be equal to the same real circuit for other frequencies, especially such frequencies as are far from the calculated one. The result of the experiments depicted in Fig. 128, Chap. X., as compared with the calculated results, shows that there may in fact be considerable divergence in the two cases if the line angle is large, and the further experiments on a circuit equivalent to 40 miles of the standard cable in Chap. X. showed that the larger the angle the wider was the divergence between the real and equivalent lines at frequencies varying widely from the calculated one as judged by voice tests. Now the equivalent line resistance in the case of an equivalent T circuit depends by inspection of the formulæ in App. III. on the value of $Z_0 \tanh \frac{\theta}{2}$, with a constant leakance, whilst the resistance of the uniform line depends

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on $Z_0\theta$. If the value of θ is small $\frac{Z_0 \tanh\frac{\theta}{2}}{Z_0\theta}$ approximates in value to $\frac{Z_0\theta}{2Z_0\theta} = \frac{1}{2}$, *i.e.* there is practically no difference between $\tanh\frac{\theta}{2}$ and $\frac{\theta}{2}$. If follows that if the equivalent circuit is constructed to represent a small line angle it may be made to conform to the real circuit for any and all frequencies to any required degree of exactitude. It further follows that loaded circuits will approximate more nearly to real ones the smaller the load and the nearer the coils are placed together. This is in effect the conclusion of Pupin, as will shortly be seen. In practice it is required to place the coils as *far apart* as possible with the largest load, for obvious commercial reasons. Owing, however, to the complex frequencies of the voice the precise rule to be adopted necessitates speech tests to determine the fundamental data. When these are derived a highly useful conclusion may be deduced from them.

182. Spacing Distance of Coils in Cables; Series Loading.—To obtain an idea of the principles on which the spacing rule is based, refer to Fig. 7. It will be seen that the horizontal line is divided into wave lengths of a definite length; the wave also travels with a definite speed. Now if inductance is added to a circuit in which a current wave of a fixed frequency is being transmitted, the wave length is shortened in inverse proportion to the product $\sqrt{LC}$ and the speed of the wave is also reduced in the same ratio. If the frequency in a given circuit is doubled, the wave length is halved. (See formula 65, Chap. VIII.) It will therefore be clear that loading shortens the wave length and reduces the speed of propagation along the line—Pupin has shown that the nearer coils are spaced in a wave, the more the circuit approximates to a continuously loaded one.

Pupin's Law states that if the interval between loads be expressed by an angle α, the difference between a coil loaded line and the corresponding uniform line of the same resistance and inductance is the difference between $\sin\frac{1}{2}\alpha$ and $\frac{1}{2}\alpha$.

If coils are too widely spaced, the circuit will be subject to greater attenuation, owing to reflections from the coils reducing the wave amplitude. Further, the upper tones of the voice have a higher frequency than the lower ones, and have therefore a shorter wave length.

If the coils are spaced a given fixed distance apart, it follows that the waves corresponding to the upper notes will have a less number of coils per wave length than the lower notes. It is, therefore, such higher notes as are necessary for clear speech that determine the closeness of coil spacing. Professor Breisig has formulated a different law. He supposes the loaded circuit to be divided up into a number of parts, each part consisting of a loading coil and half the circuit length towards the next coil on each side of the first-named coil. Each of these parts is considered as an oscillating circuit which resonates at a definite frequency, and this frequency varies inversely as $\sqrt{LC}$ varies. (See formula 65, Chap. VIII.) The spacing must be such as to keep the loaded circuit far from the

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resonating point, any near approach to it increases the attenuation.

The factor $\sqrt{LC}$ enters into all the various rules found for the spacing distance of coils, whether we consider the velocity, the number of coils per wave length. or the oscillating circuit. It may then be stated as a general rule that spacing distance varies as $\frac{1}{\sqrt{LC}}$ but to give the rule definiteness, experiments are necessary owing to the wide range of frequencies in the voice.

The English rule found by experiment is

$$CLD = 25 \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where C is the capacity in microfarads of one mile loop of the circuit, L is the inductance of the loading coil in millihenrys, and D the distance apart of the coils in miles. The product CLD must not exceed 25. This rule is in accordance with the general rule already quoted.

In addition to the distortion and increased attenuation caused by spacing loading coils too wide apart, the effective resistance of loading coils (see para. 143) increases with frequency, and the effective leakance (see para. 149) also increases with frequency. These two factors may cause a marked proportional increase in both distortion and attenuation. Both the effects of effective resistance and leakance, however, *increase* as coils are spaced more closely if formula (17) is fulfilled, *i.e.* for light loading the effects are less than for heavy loading.

In the case of the loading described in para. 189, calculation of the attenuation on the assumption of uniform distribution and lumped loading respectively for frequencies between 400 and 1200 cycles, does not show a marked difference. The impedance is, however, much more sensitive to change of frequency. In view of the fact that the theoretical considerations on which the spacing rule is based do not take into account either effective resistance or leakance, as affected by frequency, it is clear that the law cannot be regarded as absolute. It should be borne in mind, however, that formula (17) is based on practical experiments. The results may therefore be assumed to hold good in the conditions in which the tests were made. In England up to the present light loading has been largely used, and the speech is sufficiently clear. The results arrived at in various countries differ somewhat as regards the spacing rule. The number of coils and other details for the same loaded line in Great Britain, America, and Germany are as given in the following table.

Name of country.	Relative number of loading coils used.	Number of coils struck by wave front per second.	Frequency of oscillations per loading coil section.
Great Britain	100	6325	12,650
America	116	7000	14,000
Germany	126·5	8000	16,000

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See an article on the "Uniform Design of Loaded Lines," by Prof. F. Bresig, *E.T.Z.*, June 17, 1915.

It appears from published results that the loading has generally been heavier in America and Germany than in England, and this may to some extent account for the difference. Personal judgment, however, enters into the matter, and the difference of language may have some effect. It is possible that heavier loading will in future be resorted to in Great Britain, but the type of future loading is closely bound up with telephonic relay developments (see Chap. XIV.). The closeness of spacing also affects the uniformity of the impedance of loaded lines when measured at different frequencies, and this is of importance in relay working.

It is not always practicable in commercial loading to space coils according to the rule in (17), and at the same time to divide the circuit into an equal number of half loading sections as we have previously assumed. The half end sections are frequently longer or shorter than the theoretical length of the half section. It is theoretically advantageous to divide the circuit into an equal number of symmetrical sections, but the effect of this on the ultimate efficiency and cost should be carefully weighed before departing from the usual rule. (For considerations of cost, see Chap. XII.)

Loading Coils for Underground and Aerial Circuits.

183. Loading Coils for Underground Circuits.—The function of a loading coil is, of course, to add inductance to a telephone circuit; but in this case many other qualities are also desirable, the most important being —

(1) Low resistance of the coil, varying as little as possible through a wide range of frequencies, and with change of current strength.
(2) Permanent retention of the inductance of the coil.
(3) High insulation.
(4) Small electrostatic capacity.
(5) Non-interference with other coils in proximity.
(6) Economy in space design.
(7) Reasonable cost.

Let us briefly glance at the various points named.

(1) The coils used up to the present for underground work have a value of $\frac{K_1}{L_1}=50$, or 50 ohms per henry when $2\pi f = 5000$, whereas when $2\pi f = 1780$, at which frequency the coils are measured, the resistance is a little more than doubled. The reasons for the increase in resistance are given under para. 143 in Chap. VIII. A notable increase of resistance occurs with increase of current. The normal testing current is 0·8 milliampere, but with much greater current strengths the resistance may increase fourfold. Now it is highly important in telephone relay working that the so-called circuit constants should maintain a fixed value on account of the fact that this method of working requires the line to be balanced for all frequencies and current variations by means of an artificial cable which imitates the real line as nearly as possible (see Chap. XIV.). A new form of coil has recently been designed in which the constants change much less, with variations of current and frequency.

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This should effect a material improvement. (See Western Electric Co.'s British Patent Specifications, 103188/15, 107007/17, and 124658/19.) The Post Office is trying these coils.

Loading coils of different inductances, but of approximately similar size and price (for side circuits and phantom circuits respectively, the two series not being the same), can be so manufactured that the effective resistance of a coil at a given frequency is directly proportional to the inductance of the coil. That is to say, if the effective resistance is R_1 and the inductance L_1, then $\frac{R_1}{L_1}$ is a constant quantity, *i.e.*, if R_1 is 50 (say) when $L_1 = 1$, then $\frac{R_1}{L_1} = 50$. From this relationship, if L_1 is known, then R_1 can be found. These facts facilitate calculation and design, because if the effective resistance of a coil having 1 henry is known, the resistance of any other value of inductance is also known. In the case of the 300 lbs. conductors in the London-Birmingham cable, however, the coils have a maximum effective resistance of 35 ohms per henry. Coils suitable for underground work may at present be obtained with an effective resistance which does not exceed 30 ohms per henry.

(2) The inductance of the coils now used in our underground circuits is liable to be seriously reduced if direct currents of more than 100 milliampères are passed through them. The coils, in fact, become magnetized. It is possible to partly demagnetize them by sending alternating current of gradually diminishing intensity through them until the current is very small. In practice the telephone machine ringing current with a rheostat in series is effective, applying a current at the commencement not exceeding say ½ ampere and gradually adding resistance to the circuit until the current is very small. It is not, however, at present possible to completely restore the lost inductance within, say, 15 per cent. of its original value. If excessive alternating currents are applied to coils which have their full efficiency, they may suffer damage. The demagnetizing treatment described should only be applied when the condition of the coils is definitely known, and then only by skilled testing officers acting under instructions. When testing loading coils during the laying of cables, etc., a sufficient resistance should always be in the circuit to keep the current at a low figure. The new coils are very much less affected by current.

(3) The insulation between the windings of the coils is guaranteed not to be less than 1000 megohms when tested with 100 volts D.C. In practice, 10,000 megohms is usually obtained. As the coils are not guaranteed to stand high voltages, these should only be used under instructions. The maintenance of high insulation is, however, of great importance, and it is sometimes necessary to use 500 volt meggers to obtain a reliable insulation reading. Fig. 91 shows the theoretical wiring diagram of a side circuit and also a phantom circuit coil in a loading pot. The internal wiring is very complicated, and these pots should, in case of internal faults, usually be returned to the manufacturer and not opened on the spot.

(4) The capacity between the wires of loading coils is small. The capacity between the line windings of the coil most frequently used in England, must not exceed 0·0035 m.f.

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(5) When a loading coil is placed in a telephonic circuit it must be arranged so as not to disturb the balance of the electrical constants in the A and B lines. For this reason it is divided into two halves, one of which is placed in the "A" line and the other in the "B" line. (If a phantom coil is in question, each half is again divided into two halves for the same reason, as shown in Figs. 91 and 92.) The

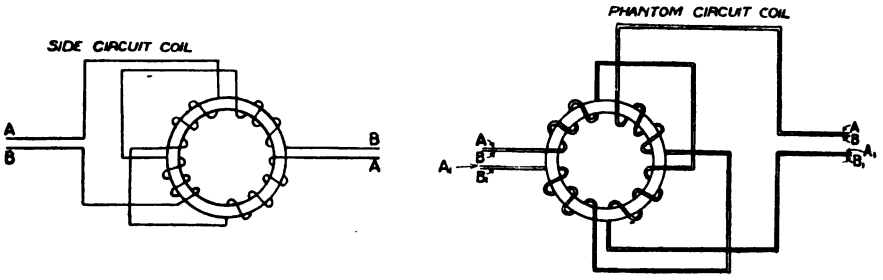

FIG. 91.

windings magnetize the coil in the same direction so that the mutual induction is added to the self-induction.

The inductance and resistance (both ohmic and effective) of the parts into which the coil is divided must be alike. As a rule a number of loading coils are assembled in the same case and much care is necessary to avoid cross-talk among them. The significance of this point may be shown by the following consideration. If the current

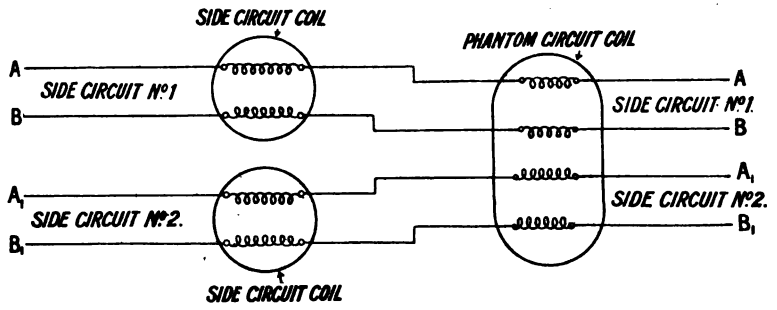

FIG. 92.

leaving the transmitter at the sending end of a line be represented by 100, we know that commercial conversation can be conducted over a line of such a length that only 1 per cent. of the outgoing current is received. It is, therefore, evident that the cross-talk must be far less than this 1 per cent.—as a matter of fact, precautions are taken in loading coil pots to ensure that it is never more than 1 per cent. of 1 per cent., even in the case of phantom circuits, or expressing the result in millionths of the outgoing current, we may say that the cross-talk between any two circuits in a pot of loading

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coils is kept below 100 millionths of the outgoing current. Now the outgoing current at the terminals of the secondary of the induction coil rarely attains 10 milliampères, so that the cross-talk is kept below 100 millionths of 10 milliampères or one millionth of an ampère. It is found experimentally that $\frac{1}{10}$th of this cross-talk can be heard in a quiet room with a good Bell receiver, *i.e.* it is possible to hear the acoustic effect of one ten-millionth of an ampère in a good commercial receiver.

(6) The loading coils used in British underground circuits are of toroidal form with a closed iron circuit and only a very small external field (which permits of their being placed near together). The coils are usually supplied in iron cases varying in size as required. The ring-like loading coils are threaded on dowels (see Fig. 93) and separated from each other by iron divisions to reduce cross-talk to a minimum. The number of dowels in a case varies according to requirements. When all the coils are in a position the case is filled with an insulating compound and carefully sealed. The coils are led out by two multiple twin cable stubs for the up and down sides of the line respectively. Until recently, only one leading out cable was used in this country, and it is still used in America. Fig. 94 shows the latest type of loading coil pot and cable as used in the United Kingdom.

In order to facilitate identification, a colour scheme is employed. Cases containing 98 coils are obtainable, but the largest loading coil cases used up to the present in this country contain 64 coils, both side circuit and phantom circuit coils being included. The case is $44\frac{1}{8}$ ins. high and $25\frac{1}{4}$ ins. in diameter (max.), each case weighs 1500 lbs. These pots are not easy to handle and it is considered preferable not to exceed 50 coils in one pot. The stub consists of impregnated cable, lead covered. Great care is required in dealing with these stubs. Faults in them should be reported to ascertain whether the pots should be returned to the manufacturer. If a crack unfortunately develops which is decided to be suitable for local repair, it should be kept under close observation afterwards. Experience has shown in some cases that an apparently good insulation does not hold afterwards. With care the insulation may be maintained at a high figure for many years. This has been proved to be the case in coils recently recovered between Manchester and Liverpool, and which had been in use about eight years.

(7) The question of reasonable cost is obviously very important. The value of the coil, however, is largely dependent upon the relative cost of the copper conductors with which it is associated, seeing that both copper and coils are separately capable of improving transmission within wide limits. Each case must be taken on its merits, and the combination which is the cheapest on the whole, selected.

184. Loading Coils for Aerial Lines.—These coils are toroidal in shape and are enclosed in strong iron cases, specially adapted for attachment to an auxiliary arm, which is fixed to the pole Fig. 100. Fig. 95 shows an aerial line coil and an ordinary underground coil, both being unenclosed, the relative sizes of the coils being to scale. It will be observed that the aerial coil is much larger than the other; this is to permit of a lower effective resistance being

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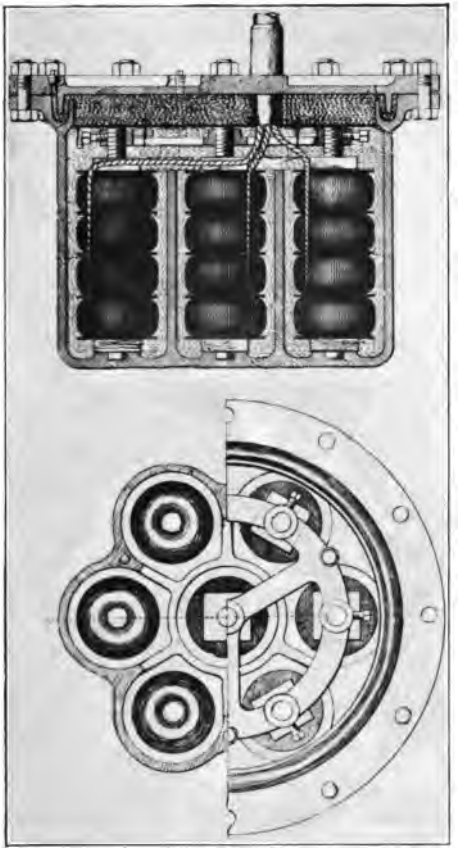

FIG. 93.—Loading Coil Case—Internal Arrangement.

FIG. 94.—Loading Coil Case with two Leading-out Cables or "Stubs."

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FIG. 95.—Relative Sizes of Loading Coils.

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FIG. 96.—Aerial Loading Coil with Leads.

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obtained, owing to the fact that aerial loaded circuits have generally a much less ohmic resistance per mile than underground circuits, and it is therefore considered necessary to keep the added resistance as low as possible. In this case the larger coil has half the effective resistance of the smaller one. Other considerations, however, require to be taken into account, and these will be dealt with later. The completed coil, ready for attachment to the poles, is shown in Fig. 96, and the lightning protector is shown in Figs. 97 and 98. It is necessary that every coil should be separately protected and that the protector should be of a reliable form and such as will stand weather exposure. The protector consists of discs of non-arcing metal and was supplied by the Western Electric Company. A complete pole fitting, as arranged by the Engineer-in-Chief of the Post Office, is shown in Fig. 99. Each loaded wire is terminated at every loading point. Each complete loading coil and case weighs

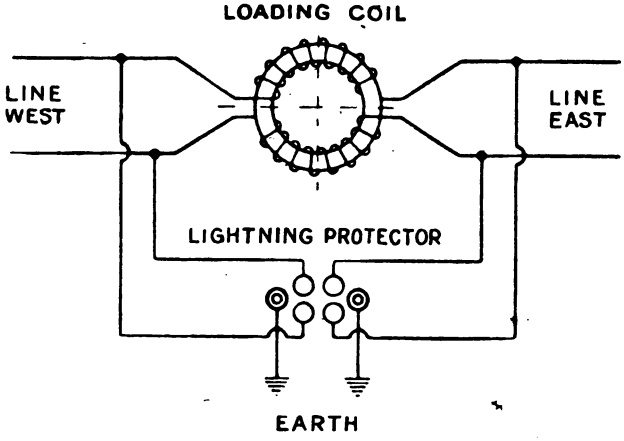

FIG. 97.—Lightning Protector for use with Loading Coils.

approximately $\frac{3}{4}$ cwt. and is fixed on an auxiliary GI support bolted to the back of the pole (Fig. 99). (The VIR leads are lead covered and the method of fitting is shown by Fig. 100.)

The whole of the underground and overhead coils used by the Post Office are made and supplied by the Western Electric Company, and the author's thanks are due to them for supplying a number of the illustrations.

The Western Electric Company have designed and manufactured for the use of the Post Office a number of aerial loading coils having an effective resistance of 35 ohms per henry $\left(\frac{R_1}{L_1} = 35\right)$ at 800 periods per second. The weight of these coils in their iron cases is two-thirds of that of the aerial coils (made by the same firm) having an effective resistance such that $\frac{R_1}{L_1} = 25$ at 800 periods per second. The coil was designed at the request of the Post Office to obtain a cheaper coil than that shown in Fig. 95,

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as experiments indicated that the new type would be sufficiently efficient for the required purpose, and subsequent experiments have fully justified its provision for loading 300 lbs. aerial lines having a leakance such that $\frac{G}{C} = 100$.

185. Factory Tests of Loading Coils.—In order to ensure as far as possible that loading coils meet the requirements for which they have been designed the following tests are applied to them at the

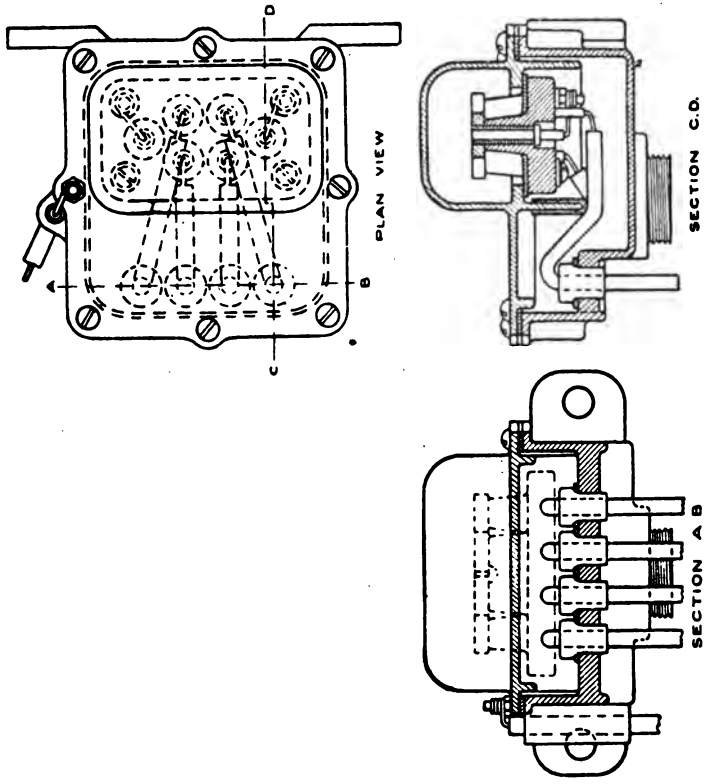

FIG. 98.—Lightning Protector.

factory by Post Office testing officers. The tests may be described under the following heads:—

Inductance balance.
Effective inductance.
Direct current resistance.
Effective resistance.
Insulation resistance.
Electrostatic capacity.
Insulation breakdown.
Cross-talk.

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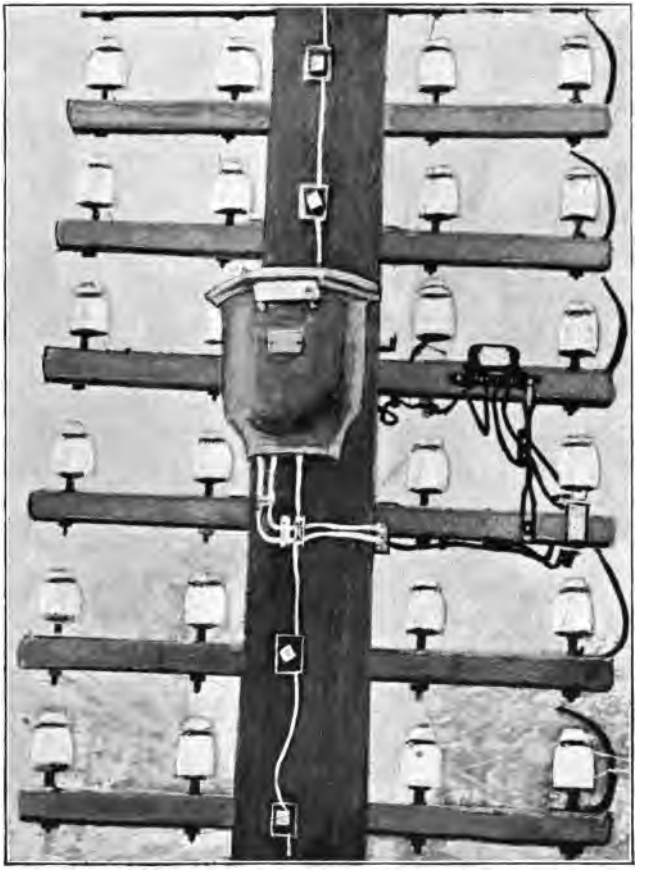

FIG. 99.—Aerial Loading Coil fitted to Pole. Protector mounted on Cross Arm.

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An alternating current Wheatstone Bridge Set, with a suitable A.C. generator, is provided by the contractors. A cross-talk meter with accessories is also provided. A general or "master" specification embodies details of the general method of test and describes

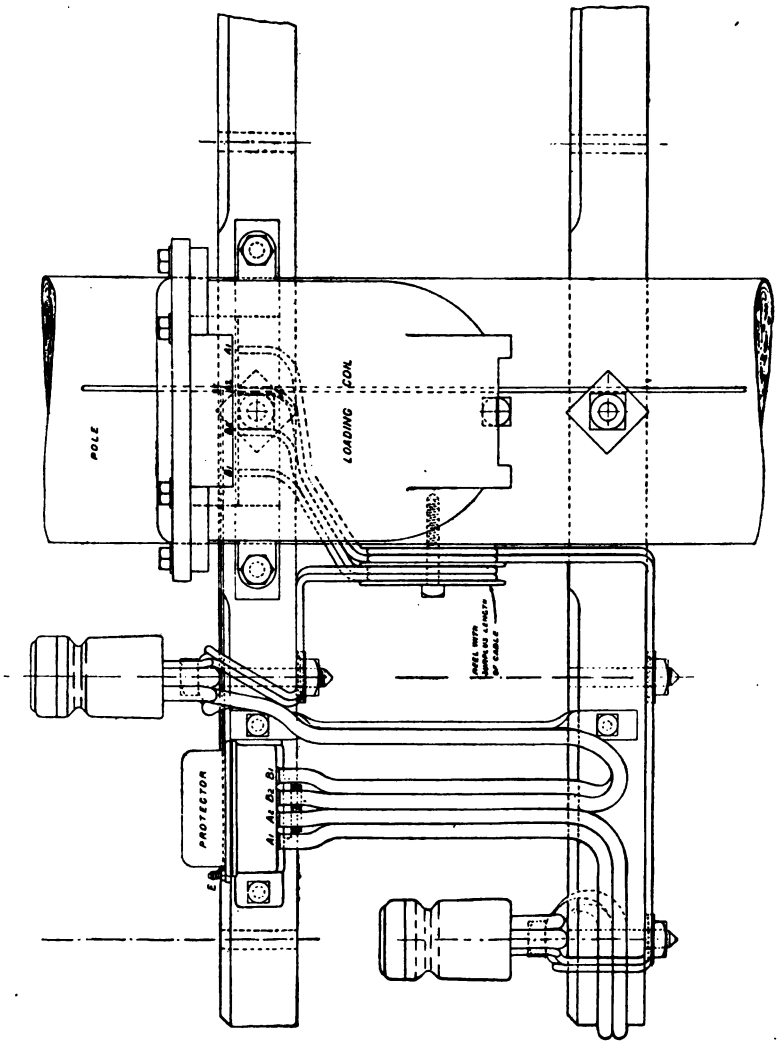

FIG. 100.—Aerial Loading Coil. Method of fixing on Pole using V.I.R. Lead-covered Leads.

MEM.—There are two insulators on each double spindle. It is only possible to show one of each in the drawing.

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the various testing devices, and a separate specification is provided in respect of each coil, which specifies the limits between which the resistance and inductance may vary, and fixes limits for the capacity insulation and cross-talk. It is not practicable to give details of all these tests, as they relate in many cases to the private methods of one firm only. The following points, however, are of interest.

The coils are divided into different categories according to the effective resistance per henry. 1780 cycles per second is the testing frequency adopted. This ensures that the coils are within known limits for the higher frequencies of the voice currents as well as for the lower ones. The manufacturers' testing current is 0·8 milli-ampère. These details apply to the older type of coil and are subject to change.

As already mentioned, the coils have an insulation not less than 1000 megohms when tested by means of a D.C. voltage of 100. An additional "breakdown" test is applied for a few seconds by means of a 500-volt generator. The overhearing is confined to coils threaded on the same dowel, and the number of millionths of the applied current which is permissible when listening by means of a telephone on neighbouring coils, depends on the number of coils on the dowel. The tests are made on the completed pots of coils and the amount of overhearing allowed is never more than 100 millionths, including the most complex phantom combination and external leads. In simpler cases it is very much less. An idea of the cross-talk testing arrangement may be gathered from Fig. 133, and the method of making cross-talk tests is also described in Chap. X.

The amount of out-of-balance of resistance and inductance allowed depend upon whether the coil is to be used for phantom working or not. If it is to be so used the requirement is more severe, but the coil is more expensive. The D.C. resistance of a coil for "medium" loading is 4 ohms.

Proposals for reducing the Number of Coils for Phantom Working.

In phantom circuit working advantage is taken of the fact that currents circulating in series in loading coil and transformer windings may be arranged to produce a flux in the iron of the coils, whilst currents circulating in the same coils in parallel can be arranged to have no magnetic flux (see Fig. 101). Loading coils are now designed on an analogous principle; two side circuit coils producing a combined flux which loads a phantom circuit without interfering with the side circuit and without the necessity of providing a special coil for the phantom circuit. (See British Patent Specifications 17615, 1915, and 110578, 1917, granted to C. E. Hay and H. W. Sullivan, London; British Specifications 8185, 22133, and 22270, 1914, granted to H. B. M. Pleijel and A. H. Olsson, Sweden; British Specification 22554, 1914, British Insulated and Helsby Cables, W. P. Fuller and H. H. Harrison; British Specification 110578; British Specifications 115685, 1918, and 115895, 1918, British Insulated Helsby Cables, G. E. Bairstow and H. H. Harrison.)

186. The Effects of Loading on Underground Air-Space Cables.—An underground air-space main line cable usually contains a large

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number of pairs—the new London-Birmingham Cable contains 52 pairs. The pairs are divided into groups of two, which are twisted together to form a " quad " made up in the so-called multiple twin formation. On the two pairs of each quad a third circuit may be super-imposed or phantomed. In such a cable there are thus a large number of circuits in close proximity. To make the best use of the cable, the circuits are required to be as efficient as possible with a minimum amount of disturbance among them, and at a minimum cost, due regard being given to economy of space and mechanical stability. So long as the circuits are unloaded, their speaking efficiency and the clearness of speech upon them leaves much to be desired. If the cables are carefully manufactured and the circuits in them unloaded, however, it is usually possible to keep cross-talk among them within small limits, without special balancing after laying. When the circuits are loaded, the efficiency of each circuit is considerably increased, but the cross-talk is much more pronounced, and phantom working is now quite impossible without special balancing of the cable pairs as regards electrostatic capacity after laying. It appears to be worth while to inquire how this comes about.

By the operation of loading (1) The impedance of the circuits has been considerably increased, so that for a given consumption of energy, the voltage in the circuit is now higher and will produce more powerful disturbing effects.

(2) The loading coils added, although balanced as well as possible, are not perfectly free from cross-talk and add a little to the disturbance in the circuit.

(3) The effects of leakance on the attenuation constant are now much increased, as already shown in para. 172, and if the leakance, resistance, and capacity are not evenly distributed on each of the wires, there will be unequal impedance and want of balance on that account. These effects will, moreover, vary with the frequency.

(4) Each of the circuits, owing to the fact that it is loaded, is now much more efficient for speech and also for the transmission of disturbances.

From these considerations it will be evident that increased cross-talk in loaded circuits may be expected, and hence the necessity from all these considerations for balancing.

Referring once more to the question of leakance, this time as regards its bearing on speech efficiency: as already shown the apparent insulation of circuits in a paper cable is enormously greater when measured by direct currents than when measured by alternating currents at telephonic speech frequencies (see para. 168). In fact, just as we have an effective resistance with alternating currents, in loading coils, which is greater than that obtained with direct currents, so we have an effective leakance which is very much greater than that obtained with direct currents, but from a different cause.

It will, therefore, readily be understood that high and uniform insulation in loaded cables is of capital importance. This is now understood by English manufacturers, and special care is given by the best manufacturers to the production of high insulation, and also

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to the accurate twisting of wires and the quality and uniformity of the materials used. The special attention now given to cables results in greatly increased speaking range. Up till recently the constant quantity $\frac{G}{C}$ (*i.e.* the leakance per farad at a frequency of 800 p.p.s.) was taken at about 60 to 80, including cables and local wiring in loaded underground airspace cables.[1] It is now taken at one-fourth of this latter quantity (para. 149). If a circuit is fully loaded the speaking range is increased 26 per cent. with the usual type of coil, if $\frac{G}{C} = 20$ instead of $\frac{G}{C} = 80$. This fact, combined with the cross-talk resulting from leakance, explains why special steps are taken to secure high insulation.

For the foregoing reasons the terminating points and connecting frames for loaded circuits should be of high insulation to alternating currents.

On this account, special terminating arrangements have been made in connection with the London-Birmingham-Liverpool Cable. Special transformers of high insulation are used, through which all circuits pass before being connected to the internal wiring. This arrangement also secures the maximum of freedom from noise in the circuits by preventing their direct connection to any local faulty circuits. The risk of damaging the loading coils by heavy currents is also reduced to a minimum. These advantages must be balanced against some loss of transmission efficiency in the transformers in certain cases.

187. The Loaded Superimposed or Phantom Telephone Circuit in Underground Circuits.—Phantom circuits have often previously been looked upon with more or less misgiving owing largely to the fact that if one of the side circuits in a phantom unit is out of order, the phantom necessarily fails with it. The maintenance of such circuits has, moreover, been difficult owing principally to the fact that cross-talk has been much more troublesome than in ordinary circuits. The difficulties, however, have been generally due to unstable electrical conditions (which are inseparable from open lines) and also unbalance between adjacent wires and pairs in cables. In a balanced cable of high insulation, however, the difficulties referred to are reduced to a negligible point, and a stable condition is normally obtained. Not only is this so, but a phantom circuit in an underground air space M.T. cable is actually more efficient than the side circuits associated with it. This may be deduced from formula (51) Chap. VIII., which applies to any circuit having its maximum loading, whether it be a phantom or side circuit. It is clear from this formula that the comparative efficiency of any two circuits, when only their capacity and resistance vary, is inversely proportional to $\sqrt{CR}$. Now in a phantom circuit the line resistance is only half that of the side circuit, but the capacity is never so high as twice that of the side circuit.

Two theoretical diagrams of a loaded phantom circuit are given in Fig. 101. These show the electrical actions in the side and

[1] See also an article by Prof. Breisig on "The Berlin-Rheinland Cable," in the *E.T.Z.* for 11 June, 1914, page 670.

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phantom circuits respectively. Only the coils at one loading point are shown to simplify the matter, although in practice this arrangement would be repeated at every loading point. The action in the side and phantom circuit is shown separately, but when all the circuits are working, the actions shown in the two diagrams would be superimposed without interfering with each other. It

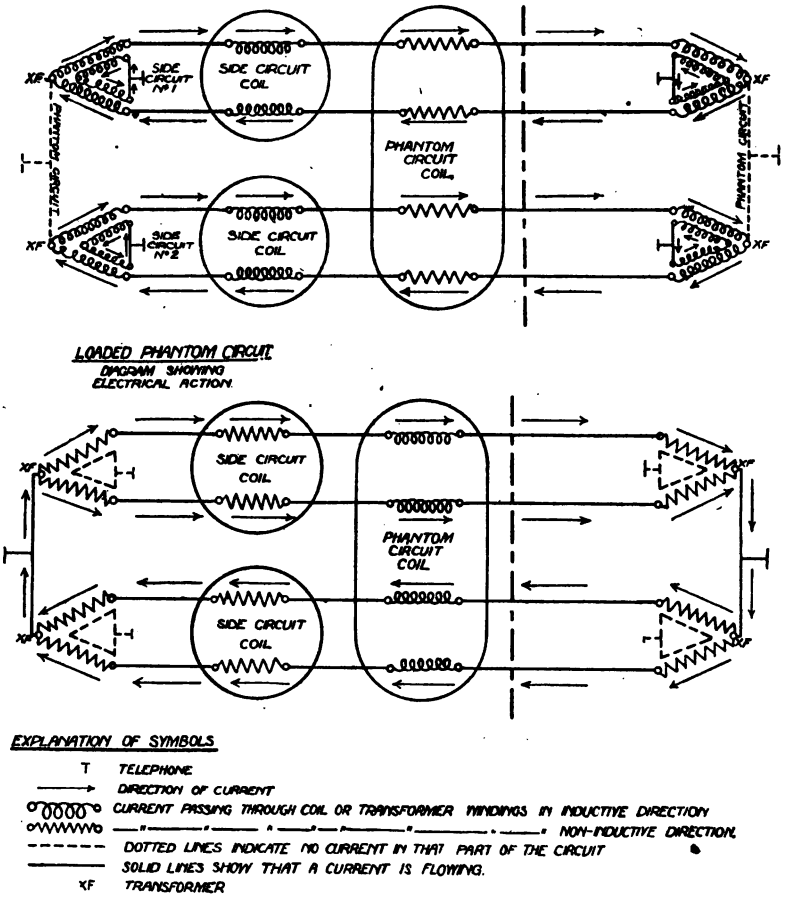
SIDE CIRCUIT COIL
SIDE CIRCUIT No 1
SIDE CIRCUIT No 2
PHANTOM CIRCUIT COIL
PHANTOM CIRCUIT
XF
LOADED PHANTOM CIRCUIT
DIAGRAM SHOWING ELECTRICAL ACTION
EXPLANATION OF SYMBOLS
T TELEPHONE
DIRECTION OF CURRENT
CURRENT PASSING THROUGH COIL OR TRANSFORMER WINDINGS IN INDUCTIVE DIRECTION
NON-INDUCTIVE DIRECTION
DOTTED LINES INDICATE NO CURRENT IN THAT PART OF THE CIRCUIT
SOLID LINES SHOW THAT A CURRENT IS FLOWING.
XF TRANSFORMER

FIG. 101.

is hoped that with the aid of the various symbols shown on the diagram it will be found self-explanatory. It will also be evident on examination that the coils in the phantom circuit in all cases add their direct current resistance to the side circuit, and that the D.C. resistance of the side circuit coils is added to the phantom circuit, so that it may be seen that although a phantom of higher efficiency

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than the side has been obtained, yet the side circuit has been somewhat reduced in efficiency. The gain in the phantom is, however, generally much greater than the loss in the side circuit.

The details in paras. 189 and 190 give an important application of the principles laid down, both as regards side circuits and phantom circuits. The type of loading employed comes under the definition of light loading.

188. Practical Applications of Series Lumped or Coil Loading to Underground Circuits.—Three stock types of coils are available for use as follows :—

		Type of Loading.
(1)	Coils having an inductance of 250 mh.	Heavy.
(2)	,, ,, ,, 175 mh.	Medium.
(3)	,, ,, ,, 133 mh.	Light.

As a general rule 250 mh. per mile is not exceeded in practical loading.

The above coils are for side circuit working. For phantom circuits the amount of inductance required is always different from that of the side circuit, as the resistance and capacity of the phantom circuit are of course different, and by (7) for the same spacing distance the value of L may be expected to be different. The reader is referred to Chap. XII. for the considerations affecting the type of loading to be chosen.

It is possible to provide phantom circuits, as already explained, but these circuits cannot at present be obtained unless the electrostatic capacity associated with each wire in a given length is equalized by selecting component shorter lengths of conductor, some of which have a capacity in excess of the mean, and others a defect, and connecting these shorter conductors in such a way that the capacity of all the conductors when measured from the ends of parts of a loading coil section are approximately equal to each other. These operations are for the most part conducted on the road during laying, but it is possible and desirable that the cable before leaving the factory should have all its conductors as well balanced for capacity as possible. The subject of balancing cables is dealt with by Mr. S. A. Pollock in articles in the *I.P.O.E.E. Journal* for April, 1914, and January, 1915, and in an Appendix to the Post Office Technical Instruction, No. XIV.

189. London-Liverpool Coil Loaded Underground Cable.—This cable was loaded by means of loading coils having a value of $\frac{R_1}{L_1} = 50$ and $\frac{R_1}{L_1} = 35$. The side circuit coils had an inductance of 133 mh., and the phantom circuit 82 mh., the spacing distance being 2·5 miles for both side and phantom circuits. The following details of Post Office tests on the completed cable are taken from the *Post Office Engineers' Journal* from an article contributed by the author.

The manufacturing, laying, and balancing of the Birmingham-Liverpool section of the cable was carried out by the British Insulated and Helsby Cables, Ltd.

The loading coils were supplied by the Western Electric Co.,

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Ltd. They were, however, fixed and jointed in the cable by the Post Office Engineering Staff.

The following table shows the mean results of direct and alternating current tests made on the cable between Birmingham and Liverpool, and gives comparative values for the efficiency of the various circuits under the heading of the Attenuation Constant β.

DIRECT AND ALTERNATING CURRENT TESTS. MEAN RESULTS PER MILE.

Cable Length = 90 *Miles.*

Weight, per mile per wire (lb.).	Diameter.		Direct current tests.				Alternating current tests ($2\pi f = 5000$).
	Inches.	Mm.	Resistance, ohms per mile (loop).	Inductance, henrys per mile (loop).	Insulation, megohms per mile (loop).	Capacity, mf. per mile (loop).	Attenuation constant β per mile (loop).
300	·137	3·48	6·28	0·058	6750	0·0545	0·00413
200	·112	2·841	9·19	0·054	17610	0·0572	0·00618
150	·097	2·461	12·44	0·052	27600	0·0685	0·00846
150	Phantom	Phantom	6·22	0·033	—	0·1008	0·00682
*100	·079	2·009	17·32	0·053	39600	0·0575	0·01014
100	Side	—	17·90	0·053	32670	0·0579	0·01057
100	Phantom	Phantom	8·95	0·034	—	0·0864	0·0837
300	,,	Not yet loaded	Not yet loaded	Not yet loaded	Not yet loaded	0·0857	—
200	,,	Do.	Do.	Do.	Do.	0·0853	—

* Not phantomed.

The mean measured characteristic impedance of the side circuits is 1060^{ω} and of the phantom circuits 585^{ω}, with a range of variation of about 8½ per cent. on each side of the mean figures.

Fig. 102 shows the effect on the attenuation constant of varying the frequency between the limits $2\pi f = 3000$ and $2\pi f = 7000$, f being the number of periods per second. The variations of β show the amount of distortion with frequency, and indicate to what extent distortion of speech may be expected. It will be seen that the range of variation is about 11 per cent. on each side of $2\pi f = 5000$. The changes are principally due to the increase in effective resistance of the loading coils, and increase of leakance in the cable, both increasing with frequency of current. The effect of coil spacing also has some effect. In practice the articulation is quite satisfactory.

Fig. 103 shows the variation of attenuation with current. This variation is probably largely caused by variations in the effective resistance of the loading coils, due to the alteration of current.

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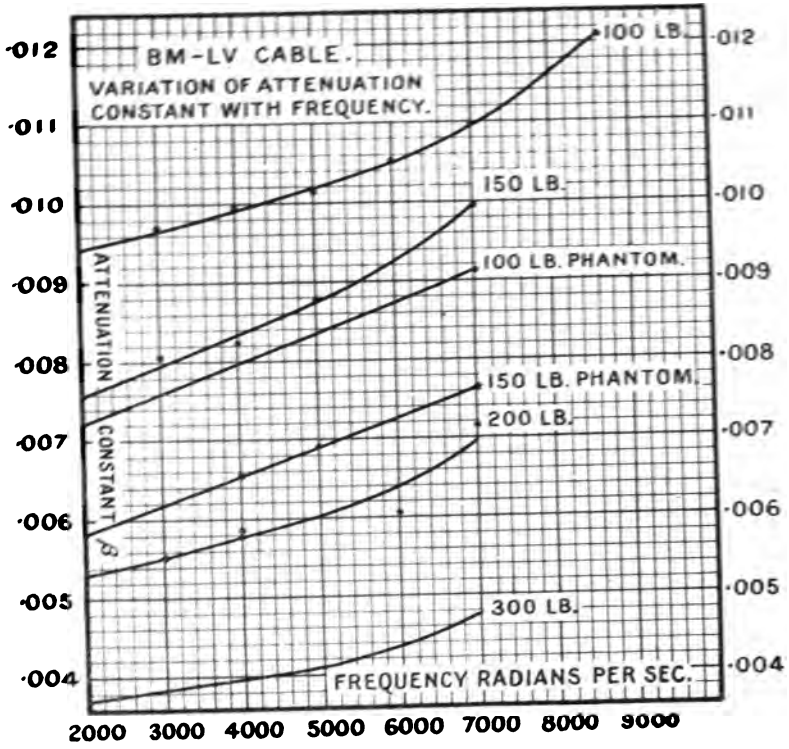

FIG. 102.

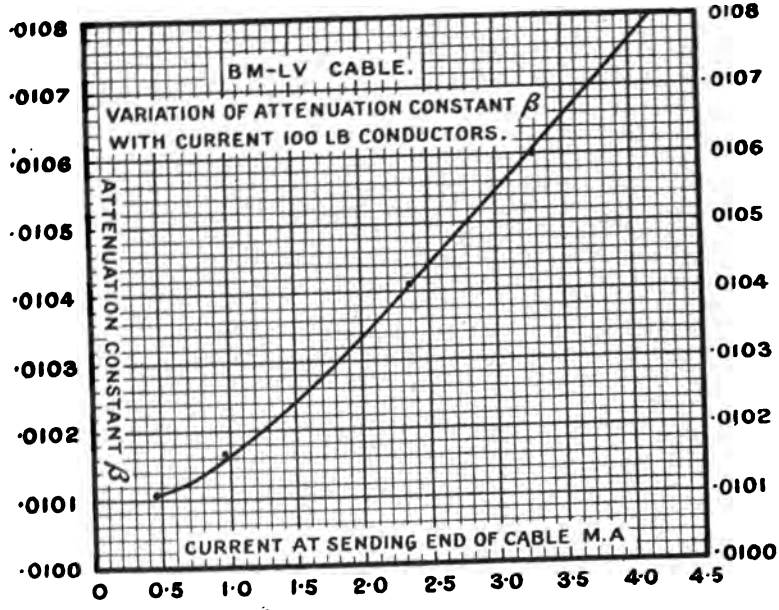

FIG. 103.

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Speech Tests.

Birmingham-Liverpool.

The following gives the result of the standard cable tests of the various types of circuits expressed in their standard cable equivalents (S.C.E.). (For details of method of test see Chap. XI.)

	Type of conductor.					
	300 lbs.	200 lbs.	150 lbs. side.	150 lbs. phantom.	100 lbs. side.	100 lbs. phantom.
S.C.E. of circuit (in miles of standard cable)	4·25	5·3	7·7	5·9	9·2	7·7

These figures do not include transformer losses (S.C.E. 1·5 per circuit) or terminal losses. For "through" connections and long subscribers' connections at the cable ends, however, these losses are generally small. In the case of short subscribers' circuits the terminal loss is greater, but the speaking efficiency on short lines is, of course, also greater than on long ones, and more than compensates for the greater terminal loss in the short lines.

190. London-Birmingham Loaded Underground Cable.—This cable was manufactured, laid, balanced and loaded by the Western Electric Company, Ltd.

The following table shows the mean results of direct and alternating current tests made by the Post Office on the cable, and gives values for the efficiency of the various circuits under the heading of the attenuation constant "β."

Direct and Alternating Current Tests. Mean Results.

Cable length = 109·5 miles.

Weight, per mile (lb.).	Diameter.		Direct current tests.				Alternating current tests ($2\pi f = 5000$).
	Inches.	Mm.	Resistance, ohms per mile (loop).	Inductance, henrys per mile (loop).	Insulation, megohms per mile (loop).	Capacity, mf. per mile (loop).	Attenuation constant β per mile (loop).
300	0·137	3·48	6·55	0·055	21800	0·0568	0·00408
200	0·112	2·841	9·62	0·054	19000	0·0655	0·00664
150	0·097	2·461	13·13	0·054	17890	0·0697	0·00884
150	Phantom	Phantom	6·56	0·035	—	0·1057	0·00756
*100	0·079	2·009	18·38	0·053	20600	0·0568	0·01074
100	Side	—	18·95	0·054	—	0·0575	0·01092
100	Phantom	Phantom	9·5	0·036	—	0·0905	0·00926
200	,,	Not yet loaded	Not yet loaded	Not yet loaded	Not yet loaded	0·0823	—
300	,,	Do.	Do.	Do.	Do.	0·0844	—

* Not phantomed.

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SPEECH TESTS (EXCLUDING TRANSFORMERS).

	Type of conductor.						
	300 lbs.	200 lbs.	150 lbs.	150 lbs. phantom.	100 lbs.	100 lbs. side.	100 lbs. phantom.
Miles of standard cable. S.C.E. of circuit deduced from speech tests .	4·7	7·4	9·5	7·9	11·3	11·6	9·4

Add 1·5 m.s.c. for Transformers.

Side Circuits—Mean Characteristic Impedance 900ω } with small negative angle.
Phantom ,, ,, ,, ,, 586ω }

Although the measured insulation wire to wire per mile was in the neighbourhood of 20,000 megohms, and even higher in some cases, the effective insulation when $2\pi f=5000$ was equivalent to about one megohm (see Chap. VIII., "Leakance").

191. The Application of Coil Loading to Aerial Lines.—Aerial circuits weighing 200 lbs. per mile and over are sufficiently loaded by their natural inductance to bring them in the category of circuits in which ωL is sufficiently great in proportion to R and ωC sufficiently great in proportion to G to permit of formula (6), App. VI., being used for calculation of their attenuation constants, and they may therefore be regarded as partly loaded. It is consequently natural to expect that loading does not produce the same degree of improvement as in the case of underground cables, where the natural inductance is relatively small. On the other hand, the degree of improvement produced by loading results in economy as compared with the alternative of increasing the gauge of copper wire. Thus by spacing coils eight miles apart between London and Glasgow in a 300 lbs. aerial circuit it can be made fully equal to a 400 lbs. unloaded circuit in the worst winter conditions; the mean annual improvement being a reduction of β by 50 per cent. As more than 400 miles of aerial circuit is involved, and the saving for double wires is 200 lbs. of copper per mile, assuming the loaded circuit to be only equal to a 400 lbs. circuit, but allowing for terminal loss, the saving in copper is obvious. It is much greater in value than the coils.

The special characteristics of aerial loading are—

(*a*) It is usually possible to load for a minimum β and still maintain a spacing distance as wide as eight miles apart in accordance with the rule given by (17). This is because of the relatively low capacity.

(*b*) The characteristic impedance is relatively high for the same reason, although the amount of inductance required to produce the minimum β is small. This is easily seen by an application of the relative formulæ, viz. $Z_0=\sqrt{\frac{L}{C}}$ and (55), Chap. VIII., to the case of a 300 lbs. conductor; the constants of this unloaded conductor are given in Table II., Chap. VIII. A series of experiments were made

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by the Post Office on this circuit, and these may be summarized as follows. The coils used have an inductance of 266 millihenrys and with eight mile spacing this gives 33·3 millihenrys per mile, to which must be added the natural inductance of the circuit 3·8 millihenrys per mile, and we thus obtain 37 millihenrys per mile. The amount of inductance required for β_m by (55) is 42 millihenrys. The inductance selected gives an attenuation constant which is within 1 per cent. of that value. The characteristic impedance is $1900\angle 0°\,24'$ with the loading named, whereas the same circuit unloaded has a value such that $Z_0=637\angle 8°\,55'$. The terminal loss with zero loop on the standard apparatus was 8·4 m.s.c. (see Fig. 144). The calculated attenuation constants, showing variation with leakance for the 300 lbs. circuit when unloaded and loaded, are shown in Fig. 104.

The mean measured attenuation constant was 0·00291 when loaded with the 266 millihenry coil and 0·00570 unloaded, at a frequency such that $2\pi f=5000$. The attenuation constant was therefore practically halved by loading. The mean value of $\frac{G}{C}$ was found to be about 135, giving two-thirds of a megohm per mile. The fact was clearly brought out in these experiments that the efficiency of loaded aerial lines depends largely on the insulation which can be maintained, and it is, moreover, the winter condition which determines the type of loading, as it is necessary to maintain the public service when the transmission value of the circuit is at its minimum (winter) point and to select the loading so as to give the best result when this occurs.

It is usually assumed in calculations that leakance is uniformly distributed, but as a matter of fact this is frequently not the case in long aerial lines, principally owing to local weather variations and leading in at testing offices. For this reason measured results may differ to some extent from calculated ones. On the whole, the variations noticed between calculation and measurement were not more than would be accounted for by local variations of insulation. The assumption of uniform leakance gives rise, on the whole, to less error than any other.

The marked terminal losses found in connection with aerial loading point to the conclusion that terminal transformers are desirable. These transformers are so designed that the impedance of the two windings of the transformer are suitable for working in connection with loaded and unloaded circuits respectively; they can, of course, be varied to meet different requirements (see Fig. 145). Aerial loading is not suitable for short lines on account of terminal losses. Its scope in the United Kingdom is limited, and moreover the present trend of progress is in the direction of loaded underground circuits. For fuller details of this type of loading see a paper by the author, "The Loading of Aerial Lines and their Electrical Constants," No. 54, Institution of Post Office Electrical Engineers, Professional Papers.

192. Attenuation Measurements of Power and Current in Coil Loaded Circuits.—The usual method of determining the attenuation constant is to deduce it from measurements of open and closed

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impedance as described in Chapter X. The formulæ on which the measurements are based are strictly applicable to uniform lines only.

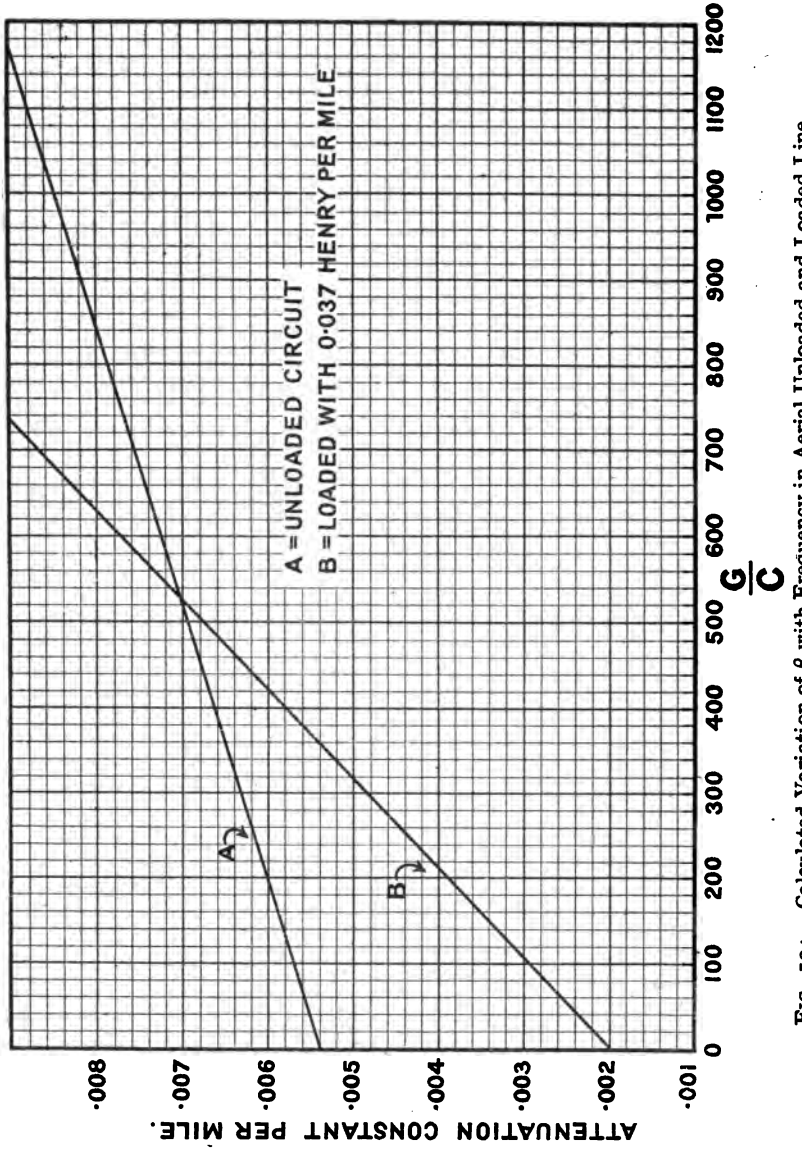

FIG. 104.—Calculated Variation of β with Frequency in Aerial Unloaded and Loaded Line.

In that case there is a fixed angle between volts and current in an infinite line, and therefore in measuring the attenuation of current

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on such a line the power can be readily deduced. Experiments have been made by the Post Office Research Section on lumped loaded lines, by measuring the power at different points on the line, and it is found that the measured attenuation of power is very near to the results deduced from measurements by the open and closed method. This latter convenient method is therefore accepted as valid, and attenuation tests are usually made on the open and closed method.

193. Distortion in Series Coil Loaded Circuits.—It follows from Pupin's rule and applications of the T and π equivalent circuit to the case of lumped loading, that attenuation increases with frequency and in the case of very high frequencies the attenuation attains excessive values.

The effective resistance of loading coils also increases with frequency. In the case of underground coils of the older type between the limits of 800 and 1780 cycles per second, the resistance of the coils is a little more than doubled, so that it may be stated that within this range the total effective resistance does not increase more rapidly than the frequency. (The total effective resistance includes the direct current and A.C. components.) The component due to frequency effects increases, however, more rapidly than the frequency, but less than the square of the frequency in the loading coils in question. In the newer type of coil the distortion should be less (see para 183). In air space lead-covered cables, experiment shows the leakance to increase directly as the frequency, the ratio $\frac{G}{\omega C}$ being constant (see para 149).

In practice excessive distortion is provided against in Great Britain by controlling within fixed limits each of the factors giving rise to distortion, thus: The error due to lumping is controlled by the spacing law (17) based on experiment. The effective resistance is controlled by specifying limits for the resistance of the coil at 1780 periods per second. The leakance is reduced as far as possible by providing for a very high direct current insulation. A selection of the most suitable materials is however essential to the best results, and this factor is borne in mind.

194. General Notes on Distortion in Transmission Lines.—The distortion on a short transmission line is materially different from that on a long one. As an example, suppose that a transmission line has two frequencies (with the same alternating voltage) impressed upon it, the frequencies being within the range of the voice, and the attenuation constants β and β_1 being $\beta = 0{\cdot}1$ and $\beta_1 = 0{\cdot}125$. The relative current value at the end of 1 mile from the sending end in an infinite line would be $\frac{\epsilon^{-0{\cdot}125}}{\epsilon^{-0{\cdot}1}} = \frac{0{\cdot}882}{0{\cdot}905}$, or a difference of 2·5 per cent. on 0·905. If the length be now increased to 30 miles the relative currents (assuming equal characteristic impedances) would be $\frac{\epsilon^{-3{\cdot}75}}{\epsilon^{-3{\cdot}0}} = \frac{0{\cdot}024}{0{\cdot}050}$, or a difference of 52 per cent. on 0·05. This shows an enormous increase in distortion with length, and experiments by means of speech tests prove that a great difference exists in distortion on very short and very long lines, in the cases where the distortion is most marked in transmission

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lines, *i.e.* high resistance underground cables and submarine cables. The effect of length should always be borne in mind in deciding the suitability of a given type of loading. In making speech transmission tests to ascertain the effects of distortion it is important that the telephone used should itself be normal from the point of view of distortion, as determined by the quality of its articulation, otherwise misleading results may be arrived at. For the same reason, before deciding the suitability of any line, or apparatus, or a combination of the two, for commerical service, it is important that the possible combinations in service conditions should be taken into account, thus loaded and unloaded lines, worked by means of telephone relays, and telephone apparatus, are all subject to more or less distortion in certain conditions, and the service can only be satisfactory when the combined effect of the whole is satisfactory. It is clear from this that tests of telephones for acceptance (for example) should take into account the line they will be required to work on, or in default of this be suitable for working on the most unfavourable combination.

195. The effect on the Attenuation Constant of the addition of the Phantom Circuit Coil.—For any particular case the attenuation constant may be arrived at by adding the resistance of the extra coil to the value of R in the formula, as applied either to a side or phantom circuit. There is, however, no necessary rigid general relation between the β or the minimum β of the side and phantom circuits. This is evident from the fact (*a*) that the capacity in aerial and underground phantom circuits has not a fixed ratio to that of the relative side circuits; (*b*) that in all classes of loaded circuits a different type of coil may be used for the side and phantom circuit. Experience shows that in the type of loading hitherto adopted in underground circuits, the addition of the phantom coil increases the β of the side circuit by 5 to 6 per cent., but this depends on the grade of the phantom coil used and the circuit in which it is utilized. In this case the side and phantom coils have the same value of $\frac{R_1}{L_1}$, *i.e.* 50.

It should be borne in mind, when calculating the minimum attenuation constant of a loaded side circuit which has phantom coil windings in its circuit, that the phantom coil direct current resistance varies with the amount of inductance, and that therefore it cannot be taken as a constant. The amount of variation of resistance depends on the method of winding the coil. In the absence of information it is sometimes taken to vary as the square root of the inductance of the phantom circuit. The effect of this increase in resistance is to reduce the maximum inductance permissible in the side circuit, as compared with the amount permissible in its absence.

196. The Application of Continuous Loading to Underground Circuits.—As already mentioned this type of loading finds a useful application in providing short lengths of cable to replace unloaded underground sections of small length in aerial lines.

The following details refer to cable of this type used in the United Kingdom and manufactured by the British Insulated and Helsby Cables, Ltd.

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The cable, which contains 24 pairs of 150 lbs. telephone conductors, is paper insulated and lead covered. The loading consists of two iron tapes wound spirally on the copper conductor in two layers.

The following are the constants of the phantom and side circuits per mile loop as measured at a frequency such that $2\pi f = 5000$:—

	Phantom circuit.	Side circuit.
$R_{eff.}$ . . .	7·05 ohms.	14·23 ohms.
G . . .	1·733 micromhos.	1·214 micromhos.
C . . .	$0·088 \times 10^{-6}$ farads.	$0·055 \times 10^{-6}$ farads.
L . . .	0·0104 henry.	0·0203 henry.
Z_0 . . .	343.	$590\angle 2°$.

Fig. 105 shows the attenuation constant of both the side and phantom circuits for different frequencies. At the standard frequency of 800 p.p.s. the result is good for a tape of the thickness used. It was found that although the attenuation constant varies appreciably with increase of frequency, the characteristic impedance remained nearly constant over a wide range of frequencies. This is probably because the increase in resistance is neutralized in the impedance measurement by the increase in leakance. The constant impedance would be advantageous in the balancing of telephonic circuits. The effective resistance and leakance, however, have no necessary connection with each other.

The ratio $\frac{G}{C}$, as measured at different frequencies, followed a straight-line law. The value at $2\pi f = 5000$ was 20.

197. The Application of Loading to Submarine Cables.—The loading of these cables is at present effected either by lumped series loading or continuous loading.

The submarine cable differs from the underground cable electrically by its relatively high capacity, this being dependent on the specific inductive capacity of the insulating medium, which is, of course usually some form of guttapercha. Owing to the expensive nature of this material it is a very important factor in determining the economical design of the cable.

Series Lump-loaded Submarine Cables.—The insertion, laying, and maintenance of loading coils in these cables present obvious difficulties which have, however, to a large extent been overcome. From the efficiency point of view the dimensions of the coils are limited largely by difficulties in laying, whilst from maintenance considerations the coils are fixed at uniform distances of one naut apart at present. The fundamental formulæ already developed in Chap. VIII., (48) to (52) and (61) to (65), and also (17) in this chapter, apply to submarine cables.

Owing to the high capacity of guttapercha covered cables the characteristic impedance of circuits in them have a comparatively low value as compared with loaded circuits in aerial and underground circuits.

Maximum speed of Signalling and Minimum β with a Given Diameter D.—Lord Kelvin was the first to formulate a law connecting the quantity of materials (copper and gutta) with the speed of

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delivering signals at the receiving end of the circuit. This law states that if a conductor of diameter d is covered by an insulating material to an outer diameter D, of the core, then the speed of signalling is a maximum for that diameter D when $\log_e \frac{D}{d} = \log_e \sqrt{e} = \frac{1}{2}$. This law is usually applied to telegraph cable circuits, but it has a close relation to the conditions governing the attenuation constants of

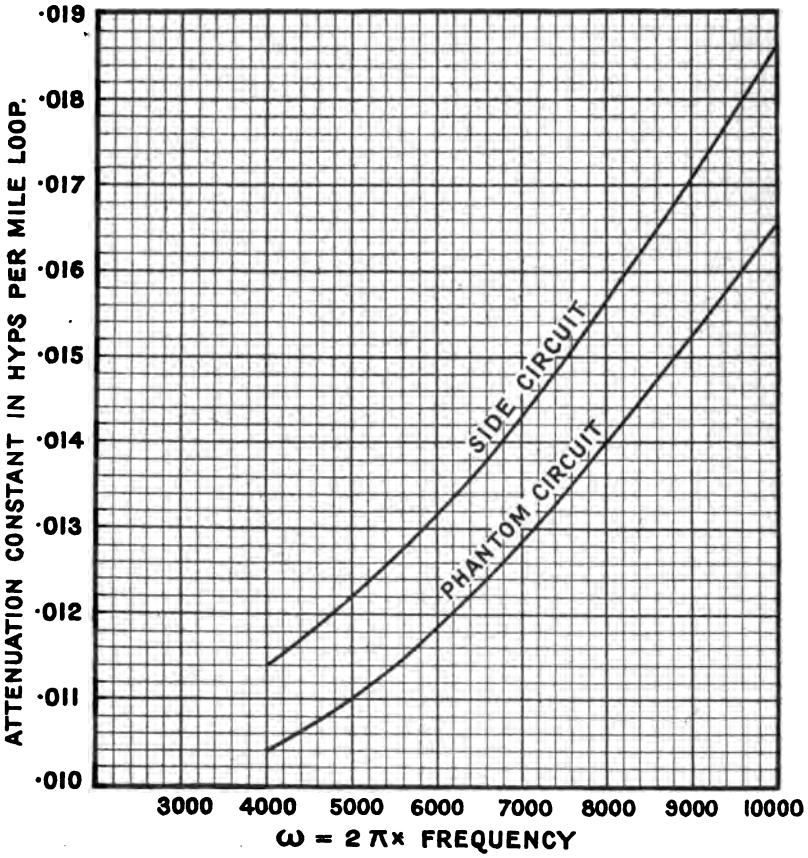

FIG. 105.—β of continuously loaded Air-Space Cable. Variation with Frequency.

telephone circuits in high resistance submarine cables, inasmuch as the conditions for maximum speed may be deduced from the formula for the attenuation constant as follows.

In the case supposed we have by (57), Chap. VIII.—

$$\beta = \sqrt{\tfrac{1}{2}\omega CR} \quad \text{or} \quad \frac{2\beta^2}{CR} = \omega = 2\pi f \quad . \quad . \quad (18)$$

whence

$$\frac{\beta^2}{\pi CR} = f \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

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In this form the frequency f (or speed) is seen to vary inversely as CR for a constant attenuation of signals β. If we write A for the constant $\frac{\beta^2}{\pi}$ the formula becomes

$$\frac{A}{CR} = f \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

But the capacity varies as $\frac{1}{\log_e \frac{D}{d}}$ by (37), Chap. VIII., and the resistance varies as $\frac{1}{d^2}$ by (13), Chap. VIII., so that the speed varies as $\frac{1}{CR}$, or $\log_e \frac{D}{d} \times d^2$. It is proved in problem 2 (96), Chap. I., that the maximum speed is obtained when $\log_e \frac{D}{d} = \frac{1}{2} = \log_e \sqrt{e}$. When this relation obtains we have

$$\frac{D}{d} = 1{\cdot}648 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

i.e. if the conductor had a diameter of 100 mils the thickness of guttapercha would only be 32·4 mils, which would not be considered to be mechanically safe, and is therefore inadmissible.

Modern telephone submarine cables are concerned with a formula of the same type as that just dealt with, but differing in details. The formula for the minimum attenuation constant in a loaded submarine cable is that given by (51), Chap. VIII., viz.

$$\beta_m = \sqrt{CR\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} \quad . \quad . \quad . \quad . \quad (22)$$

Here $\left(\frac{R_1}{L_1} + \frac{G}{C}\right)$ is a constant for a fixed frequency and may be called A_1. Formula (22) may be stated in terms of D and d as in the former case, and by analogy we may then write

$$\beta_m \text{ varies as } \frac{1}{\sqrt{\log_e \frac{D}{d} \times d^2}} = \frac{1}{d\left(\log_e \frac{D}{d}\right)^{\frac{1}{2}}} \quad . \quad . \quad (23)$$

If (23) be differentiated and the minimum value taken, we obtain

$$\log_e \frac{D}{d} = \tfrac{1}{2}$$

which is the same result as that obtained in the preceding example relating to maximum speed, except that in the case of (23) a *minimum* value is obtained, whilst in connection with (18) a *maximum* value was found. It will therefore be seen that the same conditions determine the maximum speed and the minimum β. This problem could be dealt with by plotting a graph similar to Fig. 4, Chap. I.

It is not possible to work to the ideal proportions of guttapercha and copper, for the reason given in (21), but owing to the expensive nature of guttapercha the cheapest cable is obtained for a given β when the ratio $\frac{D}{d}$ is as small as possible, consistent with

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mechanical safety. In practice the thickness of guttapercha cannot be reduced below a certain point determined by experience. The limits of thickness permitted in present practice will now be considered.

198. Formulæ for the Weight and Thickness of Guttapercha Covering of Submarine Cable Conductors.—The area of cross-section of the guttapercha with a given outer diameter of core D and a diameter of conductor d is equal to

$$\pi\frac{(D^2-d^2)}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and this area is proportional to the weight of guttapercha in the case of two conductors of equal length and different diameters.

The following formula gives the weight W of guttapercha per naut for a 7-strand conductor, d being the diameter outside the strand (D and d are in mils):—

$$W=\frac{D^2-d^2}{480} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The diameter of a solid conductor, a 7-strand conductor, and a 10-strand conductor, in mils, may be obtained as follows:—

$$\left.\begin{array}{ll} \text{Solid conductor,} & d=7{\cdot}366\sqrt{W_1} \\ \text{7-Strand conductor,} & d=8{\cdot}4\sqrt{W_1} \\ \text{10-Strand conductor,} & d=8{\cdot}11\sqrt{W_1} \end{array}\right\} \quad . \quad . \quad . \quad (26)$$

Here d is the outer diameter of the strand and W_1 the weight per naut of the conductor. Strands over 10 in number are sometimes used, but the diameter does not vary more than 2 per cent. from the 10-strand wire in any case.

A method often employed by manufacturers is to provide guttapercha in accordance with the rule that the thickness T of guttapercha varies as the square root of the diameter d of the conductor, T and d being expressed in the same unit—mils or mm.—

$$T=a\sqrt{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where a is a factor determined by experience.

A figure recently employed by the British Post Office is

$$T=8\sqrt{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

for unloaded or coil loaded cables, and

$$T=8\sqrt{d+2t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

for continuously loaded cables, t being the thickness of the iron winding.

The 310 lbs. loaded cable in Table I. is made in accordance with (28). A near approximation to the weight in lbs. per naut based on the thicknesses given in formula (28) is obtained by the following arbitrary formula:

$$W=1{\cdot}9(d-38) \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and in the case of formula (29):

$$W_1=1{\cdot}9\{(d+2t)-38\} \quad . \quad . \quad . \quad . \quad (31)$$

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In these cases d is expressed in mils: it is the diameter outside the copper strand.

To see the effect of the addition of wire to the conductor an example will be useful.

A conductor weighing 308 lbs. per naut (75·3 kg. per Km.) has a diameter d outside the copper strand of 142 mils (3·6 mm.). From (30) the weight of gutta required is 198 lbs. per naut (48·01 kg. per Km.). If this wire were covered with an iron winding 18 mils (0·457 mm.) thick, the diameter $d + 2t$ would be 178 mils (4·52 mm.), the weight of gutta required from (31) would be 266 lbs. per naut (65·04 kg. per Km.).

199. Leakance in Guttapercha and Balata Cables.—Ordinary guttapercha has a relatively high leakance to alternating currents, the value of $\frac{G}{C}$ (see Chap. VIII., para 149) being in the neighbourhood of 120 at $2\pi f = 5000$. This has a serious effect on the attenuation constant and limits the improvement which can be obtained by loading. Messrs. Siemens Bros. & Co. in 1911 introduced a form of balata dielectric with a value of $\frac{G}{C}$ at $2\pi f = 5000$ not exceeding 20. The use of this dielectric results in a notable decrease in the attenuation constant, as may be seen by comparing the Anglo-French cable, which is insulated by guttapercha, with the later types in which the new type of dielectric is used. (See Table I.)

200. Economy of Material resulting from Loading.—It was formerly considered necessary, before the introduction of loaded cables, to employ a thick insulation of guttapercha to reduce the capacity, and consequently the distortion of speech, in unloaded cables. The usual type of cable was one having conductors weighing 160 lbs. per naut and a guttapercha dielectric weighing 300 lbs. per naut. This was also employed in the Anglo-French loaded cable shown in Table I., but the experience with that cable showed that the thickness of gutta could be reduced, and this was done in all subsequent coil-loaded cables, the weight of guttapercha being reduced to 150 lbs. per naut, the conductor remaining the same. The material reduction in leakance with the newer type of dielectric at the same time reduced the distortion. The speaking range of the same cable is increased nearly four times by coil loading. Fig. 106 shows tests of a cable of this type. The upper curves relate to side circuits and the lower one to the phantom circuit.

201. Impedance of Coil-loaded Cables.—If the impedance of these cables is measured at frequencies which increase by a small number of cycles per second, over a large range of frequencies, as shown in Fig. 107, the frequency curve is not smooth and regular, but it rises and falls as shown in the figures.

This curve represents the result of impedance measurements at frequencies varying from 480 cycles per second rising by 40 cycles to 1280 cycles per second. Similar curves are obtained as the result of tests on underground loaded cables. It is possible that the cause of this phenomenon is that the capacity of the cable, or the inductance of the coils, or both, are not absolutely uniform along different sections of the cable. Every coil with half a loading

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section on each side of it, has a tendency to resonate at some frequency, and as various frequencies are applied to the cable in the process of measurement, some of them, in combination with the capacity and inductance of the cable, give greater amplitude of sent current than others. The curve shows the resistance and reactance components of the impedance plotted separately. These variations of impedance have a special significance in telephone relay working, where it is necessary to balance the loaded circuit by means of an artificial circuit, as shown later in Chap. XIV.

Table I. shows the various types of coil-loaded submarine cables

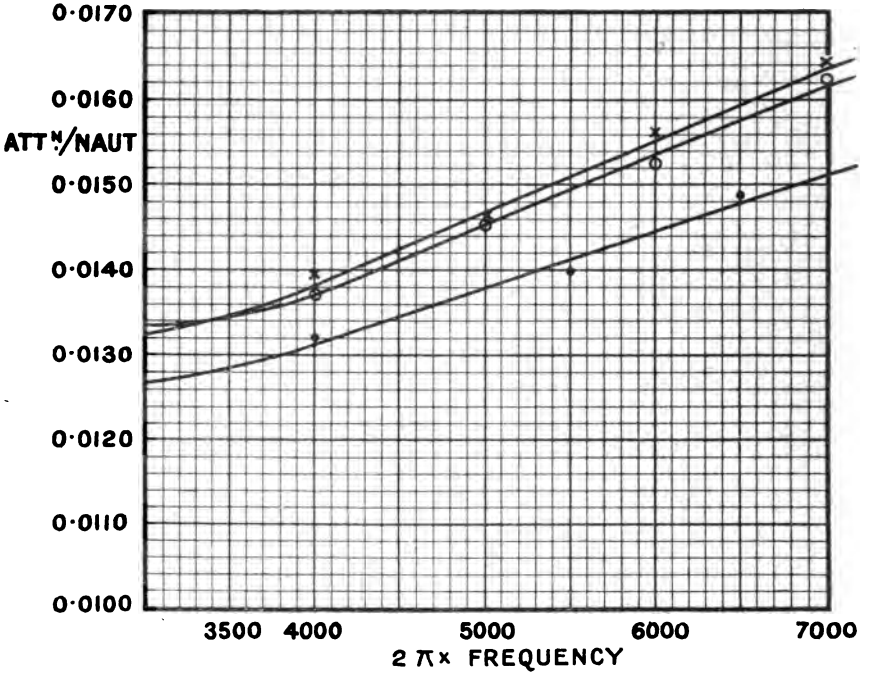

FIG. 106.—Variation of β with Frequency in a Coil-loaded Submarine Cable.

used by the British Post Office. The list is not exhaustive but it includes types of all the coil-loaded cables used.

202. Experiments in working Two Loaded Phantom Circuits in a 4-Wire Submarine Cable.—In addition to the two direct circuits formerly worked in the 4-wire Anglo-French telephone cable, two phantom circuits have also been worked in the continuously loaded submarine cable between Abbots Cliff and Grisnez. This service was proposed by the author and maintained satisfactorily for some months prior to the war. One of the phantom circuits is worked on the "loop," and the other is made up of four wires earthed at each end of the cable. Currents passing in parallel along the four wires in the same direction do not interfere with loop working

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TABLE I.

Coil-Loaded Cables. Types used by British Administration.

Title.	Where laid.	Length in nauts.	Per naut. Weight, Cu, lbs.	Per naut. Weight, G.P., lbs.	Outer diameter of Cu strand, mils.	Outside diameter of G.P. mils.	$\frac{R_1}{L_1}$	$\frac{G}{C}$	Per naut. R ohms.	Per naut. L m.h.	Per naut. C m.f.	Per naut. β	Z_0 Vector ohms.	Frequency at which measured.
Anglo-French . .	Abbots Cliff—Grisnez	20	160	300	106	—	60	120	14·2	100	0·021	0·02	1000	1000 cycles per sec.
Anglo-Belgian— Side Circuit	St. Margaret's—La Panne	47·9	160	150	106	286	60	12	14·2	100	0·157	0·0177	800	800 ,, ,,
Phantom Circuit	—	—	—	—	—	—	—	12	7·1	50	0·314	0·0173	400	800 ,, ,,
Anglo-Irish— Side Circuit	Nevin to Howth	63·4	160	150	106	286	48	15	14·2	100	0·166	0·015	$690 \angle 2° 40'$	800 ,, ,,
Phantom Circuit	—	—	—	—	—	—	48	15	7·1	50	0·320	0·015	$446 \angle 0° 52'$	800 ,, ,,
Recent type of cable Side Circuit.	—	—	310	200	142	334	*40	20	7·42	80	0·187	0·011	$647 \angle 0° 55'$	800 ,, ,,
Phantom Circuit .	—	—	—	—	—	—	40	20	3·71	40	0·378	0·0106	$343 \angle 5° 20'$	800 ,, ,,

All these cables were manufactured by Messrs. Siemens Bros. & Co., London.

* This coil is now used in cables of the Anglo-Irish type.

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either on the side currents or loop phantom circuits. This experiment definitely proves the possibility of single-wire communication on a

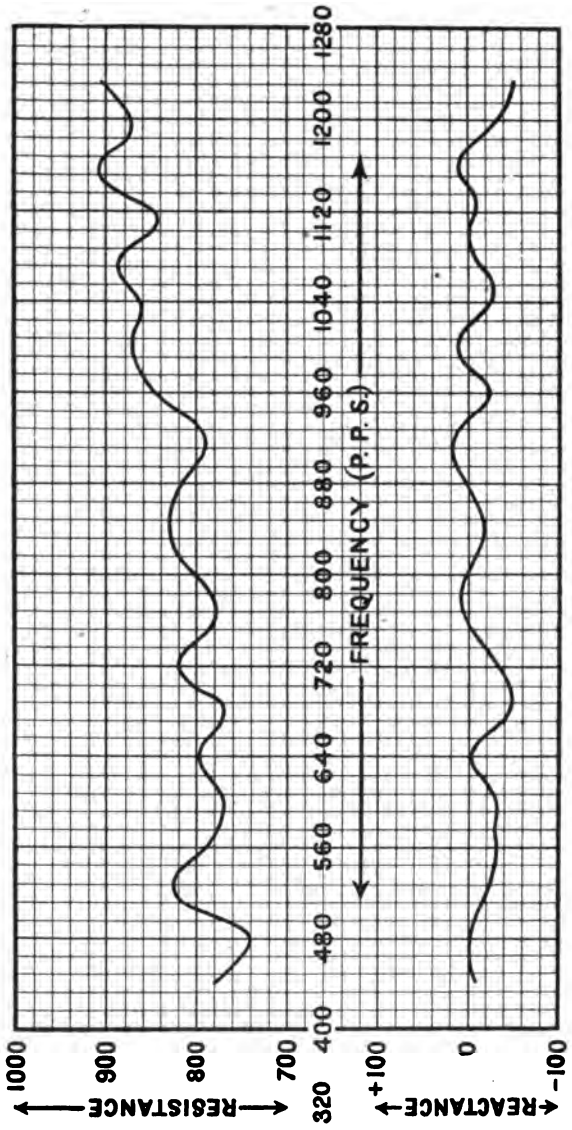

FIG. 107.—Variation of Characteristic Impedance with Frequency in Series Loaded Cables.

congested cable route, and it at the same time shows a cheaper means of providing four telephone channels of communication than would be possible in separate single-wire cables.

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It must be premised to justify this assumption that the sea is not too deep for a 4-wire cable to be satisfactorily laid. There is, of course, no difficulty in laying 4-wire cables in the English Channel, and generally round the English coast, but in deep seas there are cases where the weight of such a cable would be too great, but where a single-wire cable could be laid.

A theoretical diagram of the arrangement for working four circuits in a 4-wire coil-loaded cable is given in Fig. 108.

The arrangement of coils shown is that provided by Messrs. Siemens Bros. & Co., and used in 4-wire coil-loaded cables by the British Administration. The explanation of side and double-wire phantom working is the same as that given in connection with Fig. 101. In this case, however, the two separate coils used for phantom working theoretically permit of the four-wire earth circuit being also loaded: the currents in this circuit are shown by the straight arrows in Fig. 108. The continuously-loaded cable, however, may claim the distinction of being the first to permit of solving at the same time the problem of single-wire telephone working on a congested route, and of obtaining four telephone circuits from four wires in one cable. Up to the present it has not been found possible to successfully establish the earthed phantom on the coil-loaded cable, although in practice the double-wire phantom is quite successful in the same type of cable. The theoretical diagram given in Fig. 110 shows the four single-wire circuits extended to the distant terminal station by means of three land-line circuits, the fourth being obtained by superimposing. This is, of course, theoretically possible, but in numerous cases the aerial lines are too long or the conditions too difficult to permit of a stable superimposed balance being obtained in the aerial section.

In such a case four separate aerial circuits are provided as shown in Fig. 109. It is absolutely essential that all direct connection between the land lines and the cable be avoided. If this is done, the currents shown by arrow-heads in Fig. 110, and which are induced in the aerial telephone loops by extraneous sources of disturbance, *i.e.* telegraphs, power circuits, etc., are annulled at the centre point of the transformers on the land side. If there were metallic connection between the aerial lines and the cable, however, this would not be the case; on the contrary, the induced currents passing in parallel over the cable wires, would disturb the earthed phantom circuit and would probably render it unworkable, this was indeed proved in the preliminary experiments made before the phantom circuits were established.

The only serious difficulty in obtaining four circuits in a four-wire cable arises from the fact that the four conductors have not exactly the same electrical constants.

The distributed capacity of the four wires especially shows a want of equality which tends to give rise to inductive disturbance in a loaded cable.

The capacity of a submarine cable is very high. For example, the capacity of a single-wire to earth in a guttapercha insulated conductor of large diameter, is often more than 0·4 microfarad per naut. A small percentage difference in the capacity of two such wires joined in parallel for phantom working is sufficient to

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produce appreciable overhearing. This source of out-of-balance affects both coil-loaded and continuously-loaded cables, and satisfactory phantom working demands, in fact, much greater exactitude

TWO SIDE CIRCUITS AND TWO PHANTON CIRCUITS IN A 4-WIRE LOADED SUBMARINE CABLE.

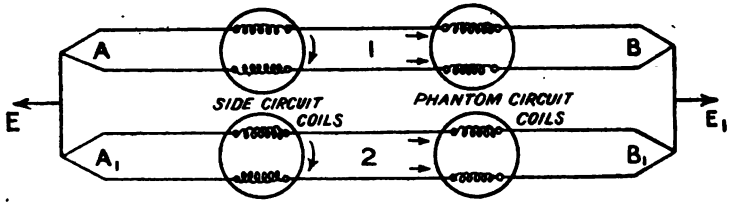

FIG. 108.—Theoretical Diagram of Cable Circuits.

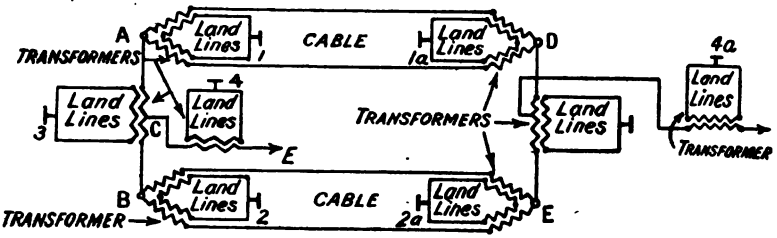

FIG. 109.—Cable connected to Four separate Land Lines at each End.

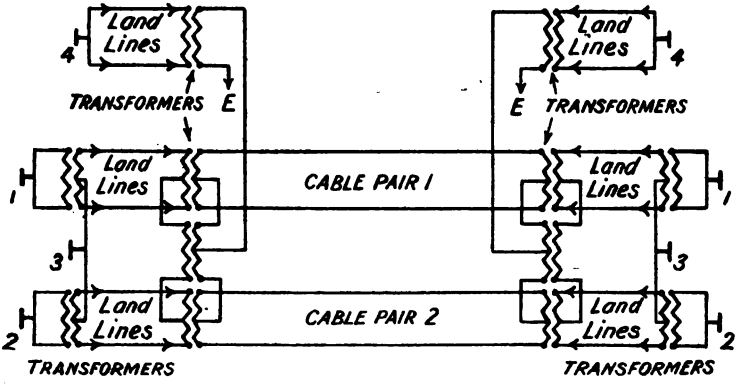

FIG. 110.—Cable connected to Three Land Lines. A Fourth being provided by a Phanton on Pairs 1 and 2.

of capacity balance than is normally obtained in manufacture. In underground cables the difficulty is overcome by crossing the wires until the conductors have a nearly equal capacity. In the case of four-wire submarine cables, however; the difficulty is greatly

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increased, owing to the relatively high capacity of such cables, and the fact that the number of possible combinations in four wires is much smaller than in underground cables. In coil-loaded submarine cables the difficulties are increased by the necessity of separate coils for the phantom circuit. If the balance difficulty can be satisfactorily dealt with, there appears to be no reason why four telephone circuits from four-wire cables should not be a permanent feature.

203. Continuously-loaded Submarine Cables.—The principal advantages and disadvantages of this type of cable have already been outlined. The extra thickness of guttapercha necessitated by this method of loading (for equal mechanical strength) may be seen by comparing (28) and (29). On the other hand, any extra cost of maintaining coil-loaded cables as compared with continuously-loaded ones should be taken into account in making a choice between them.

The possibility of obtaining a fourth or earthed phantom circuit in a four-wire cable of this type constitutes a valuable asset for it.

There are varieties of service for which this cable has great advantages. For example, where it is required to cross a wide river or an estuary with a large number of wires, it is possible to obtain a relatively large number of circuits in a continuously-loaded cable, where coils would be difficult or impossible to lay and maintain. It has, for example, been proposed, and appears to be practicable, to obtain a seven-quad continuously-loaded cable with 70 lbs. conductors and continuous loading. If, as hoped, seven phantom circuits could be added to the fourteen pairs, a very valuable cable would be obtained. Technical data of such a cable are given in Table II. In this connection it is probable that the extension of telephone relay working will render it advantageous to utilize light conductors for cross-channel work, and the ordinary four-wire cable could then be replaced by a cable with thinner conductors and more pairs of the continuously-loaded type. It must not be overlooked, however, that in the event of a break-down more circuits would be thrown out of use than formerly. On the whole, the prospects for the extension of the use of this kind of cable are very favourable.

Table II. shows details of some recent continuously-loaded cables. A complete comparison would necessitate results at several frequencies to show the relative distortion, but unfortunately these are not always available. Such comparisons are, however, very important in these cables in view of the wide range of permeability and effective resistance. To give an idea of the relative quantities of copper and guttapercha required for coil-loaded and continuously-loaded cables respectively Table III. has been prepared; this shows the result of loading a 300 lbs. conductor by both methods, the guttapercha being arranged so as to give the same mechanical strength. It is clear from the comparison that on this basis the method of continuous loading must be the more expensive of the two. This, however, is only first cost, the relative advantages and disadvantages of the two types already pointed out must also be taken into account.

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TABLE II.

TYPES OF RECENT AND PROPOSED CONTINUOUSLY LOADED CABLES.

Title.	Length in nauts.	Per naut.		Outer diameter of copper strand.	Outside diameter of G.P.	$\frac{R_1}{L_1}$	$\frac{G}{C}$	Per naut.				Z_0 vector ohms.	Frequency at which measured.
		Weight, Cu, lbs.	Weight, G.P., lbs.					R ohms.	L m.h.	C m.f.	β		
Anglo-French, Abbots Cliff—Grisnez	23·4	300	300	136 mils with wire 12 mils.	412 mils.	110	109	7·8	13·5	0·176	0·0185	$278 \angle 2°23'$	1000 cycles per sec.
Dalmatia-Istria, Adriatic [1]	—	210	180	2·76 mm. with winding ·6 mm. thick.	8·4 mm.	81	94	11·0	19·8	0·216	0·0242	$302 \angle 2°45'$	800 ,, ,,
Vancouver-Victoria	—	300	300	—	—	—	—	7·5	—	—	0·019	—	800 ,, ,,
Proposed British cable for estuary work	—	70	110	62 mils with winding, 10 mils thick.	246 mils.	60	—	34	20	0·15	0·05	—	800 ,, ,,

[1] See article in *Electrotechnik und Maschinenbau*, 6th and 13th July, 1913, by E. F. Petritsch.

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TABLE III.

SUBMARINE TELEPHONE CABLE.

Table giving comparative electrical and attenuation constants of coil- and continuously-loaded circuits having a resistance of 7·42 ohms per naut, showing variation with change of $\frac{R_1}{L_1}+\frac{G}{C}$ and with different thicknesses t of iron winding.

	Per naut.																
	Weights and dimensions in mils and lbs.					Electrical constants of looped circuits at $2\pi f = 5000$.											
	d	t	D	Weight of copper.	Weight of G.P.	R	R of phantom coil.	L added (hys.).	L nat. (hys.).	C microfarads.	$\frac{R_1}{L_1}$ of side circuit coil.	$\frac{G}{C}$	L_{max}	$\beta_m \times y = \beta$			Z_0
														β_m	y	β	
Coil-loaded cable . . .	142	—	334	308	198	7·42	1·0	0·049	0·002	0·22	40	120	0·041	0·0173	—	0·0173	430
	142	—	334	308	198	7·42	2·0	0·093	0·002	0·2	40	20	0·093	0·0109	1·0	0·0109	684
											$\frac{R_1}{L_1}$ of iron winding.						
Continuously-loaded cable	142	12	372	308	243	7·42	—	0·011	0·002	0·22	60	120	0·041	0·0171	1·17	0·02	243
(Two moderate thicknesses of wire, viz. 12	142	18	391	308	266	7·42	—	0·016	0·002	0·22	60	120	0·041	0·0171	1·08	0·0186	286
and 18 mils, have been	142	12	372	308	243	7·42	—	0·011	0·002	0·2	60	20	0·093	0·0109	1·54	0·0164	255
used as illustrations.	142	18	391	308	266	7·42	—	0·016	0·002	0·2	60	20	0·093	0·0109	1·35	0·0147	300
In practice the 18 mils would probably consist of two windings.)																	

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204. Relation between the Thickness of Loading Wire and the Attenuation Constant.—This can be shown in a general form as follows :—

Formula (12) gives the inductance L of a wire of diameter d with a given thickness of iron t. If the same conductor of diameter d be wound with a different thickness of wire t_1 to produce an inductance L_2 the value of $\frac{L}{L_2}$ is

$$\frac{L}{L_2} = \frac{10^{-4} \times 2\pi\mu \times \left(\frac{pt}{d+pt}\right)}{10^{-4} \times 2\pi\mu \times \left(\frac{pt_1}{d+pt_1}\right)}$$

$$= \frac{(d+pt_1)t}{(d+pt)t_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Having thus obtained an expression for $\frac{L}{L_2}$, next let t be the thickness of iron required to produce the value of L required for β_m, and let t_1 be the thickness of wire required to produce any less value of inductance L_2. t and d will in that case be constants and t_1 variable. Then (5) may be written

$$\left.\begin{aligned} &\frac{\sqrt{\frac{(d+pt_1)t}{(d+pt)t_1}} + \sqrt{\frac{(d+pt)t_1}{(d+pt_1)t}}}{2} = y \\ \text{and by (6)} \qquad &\beta = \beta m \times y \end{aligned}\right\} \quad . \quad . \quad (33)$$

205. Lumped Loading by Leaks (Leak Loading).—Professor Silvanus P. Thompson took out a British Patent (No. 22034, 1891) for the application of distributed inductance leaks applied to telephonic circuits, and this appears to be the first recorded definite proposal for leak loading.

A formula for the attenuation constant of the leak-loaded circuit may be derived from that for the complex attenuation constant, which applies to uniformly distributed circuits, viz.—

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad . \quad . \quad . \quad (34)$$

(See (14), Appendix I.) To adapt this formula for leak loading suppose that an impedance $R_1 + j\omega L_1$ is placed as a leak from wire to wire at sufficiently short intervals to permit of the distribution assumed in the formula still being effective. Assume that the impedance of the conductor $R + j\omega L$ is unaltered. In the stipulated conditions $G + j\omega C$ becomes $(G + j\omega C) + \frac{1}{R_1 + j\omega L_1}$

$$= G + j\omega C + \frac{R_1 - j\omega L_1}{R_1 - j\omega L_1} \times \frac{1}{R_1 + j\omega L_1} = (G + j\omega C) + \frac{R_1 - j\omega L_1}{R_1^2 + \omega^2 L_1^2}$$

$$= G + j\omega C + \frac{R_1}{R_1^2 + \omega^2 L_1^2} - \frac{j\omega L_1}{R_1^2 + \omega^2 L_1^2}$$

Here R_1 is the effective resistance of the added impedance and

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L_1 its inductance. Separating the real and imaginary quantities we have

$$G+\frac{R_1}{R_1^2+\omega^2L_1^2}+j\left(\omega C-\frac{\omega L_1}{R_1^2+\omega^2L_1^2}\right) \quad . \quad (35)$$

Substituting this value for $(G+j\omega C)$ in (34) we obtain

$$\gamma=\sqrt{(R+j\omega L)\left\{\left(G+\frac{R_1}{R_1^2+\omega^2L_1^2}\right)+j\omega\left(C-\frac{L_1}{R_1^2+\omega^2L_1^2}\right)\right\}} \quad (36)$$

This is the formula for the complex attenuation constant of a lumped leak-loaded circuit at some definite frequency.

In order to find the attenuation constant β the real part of γ is taken as in the case of series loading. It may be seen by (36) and verified by calculation that this formula is much more sensitive to changes in frequency than the corresponding formula for series lump loading: (48), Chapter VIII. For a given frequency the minimum value of (36) is

$$\gamma=\sqrt{(R+j\omega L)\left(G+\frac{R_1}{R_1^2+\omega^2L_1^2}\right)} \quad . \quad . \quad (37)$$

this is attained when $C=\frac{L_1}{R_1^2+\omega^2L_1^2}$, as may be seen by inspection of (36).

In the case where ωL is negligible by comparison with R, as in small-gauge underground circuits, this becomes

$$\gamma=\sqrt{R\left(G+\frac{R_1}{R_1^2+\omega^2L_1^2}\right)} \quad . \quad . \quad . \quad (38)$$

and as this gives the real part of γ it is equivalent to β.

If the frequency factor disappears, *i.e.* for steady currents, we have

$$\gamma=\sqrt{R\left(G+\frac{1}{R_1}\right)} \quad . \quad . \quad . \quad . \quad (39)$$

which is of the same form as (38).

It will be seen by comparing the formulæ for leak and series loading that by comparison with the ideal attenuation constant $\beta=\sqrt{RG}$ series loading theoretically increases R by the resistance of the loading coil, and leak loading increases G to $G+\frac{R_1}{R_1^2+\omega^2L_1^2}$. As a result Z_0 increases with frequency in series loading and decreases with frequency in leak loading. In practice, as already noticed, G increases in any case with alternating currents (para. 149, Chapter VIII.).

In Chapter IV. the T and π equivalent circuits are used to show the equivalence of the series-loaded to the leak-loaded circuit.

Although the leak-loaded circuit attenuation constant may be made equal to that for the series coil-loaded circuit for one frequency, yet the mean of a large number of frequencies which are much

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higher and lower than the selected one, would have a higher attenuation constant for leak loading than for series loading, owing to the tuning effect of the former, and the rapid increase of attenuation with frequency above and below the critical frequency aimed at.

As regards the spacing of coils, experiment shows the same general effects in both leak and series loading, *i.e.* if the coils are too widely spaced the articulation suffers due to this cause, and it is a matter for experiment to decide what particular spacing gives results sufficiently good for commercial service. As shown by (17) the point has been cleared up as regards series loading, but a definite and reliable experimental figure for leak loading is not known by the author. It is generally assumed that leak loading, owing to its greater wave length, would permit wider spacing; but the distortion due to this type of loading is marked, and this might influence the practical spacing distance.

Professor J. Perry, F.R.S., in a paper on "Telephone Circuits" read before the Physical Society on February 25th, 1910, adapted formula (36) to the introduction of condensers in series, by changing $R+j\omega L$ to $R+j\omega L+\frac{1}{j\omega C_1}=R+j\omega\left(L-\frac{1}{C_1\omega^2}\right)$. The effect of the

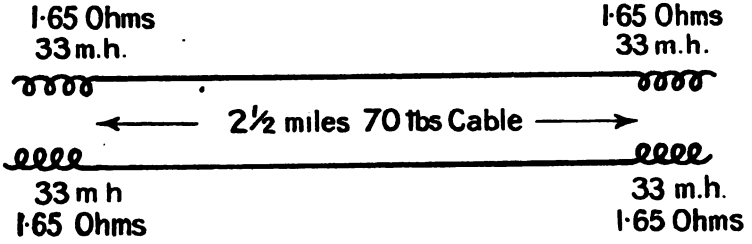

FIG. III.

capacity could be arranged so as to counteract the tuning effect of formula (36). Proposals have also been made to modify the effects of frequency by other means. See, for example, British patent specification No. 65, 1911, granted to Friedrich August Becker.

Some preliminary experiments on leak loading have been conducted by the Post Office but the results obtained were not satisfactory, the distortion being very marked, and the mean attenuation to speech currents relatively high with the type of coil available for the experiments. It is probable that specially designed coils would give better results. Difficulties in testing for faults and in signalling would, however, be experienced in this type of loading, and it has not up to the present been practically adopted.

Example 1.—A 70 lbs. underground cable circuit loop, 2½ miles in length, having the constants shown in Table II., Chap. VIII., has an inductive load added to each end of the loop in series with the conductors. The effective values of the inductance and effective resistance of the loads are as shown in Fig. III.

Construct an equivalent circuit diagram of the loaded loop.

Example 2.—Find the line angle and characteristic impedance of a uniform unloaded circuit of the same length as the loaded one

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($2\frac{1}{2}$ miles) and to which the loaded equivalent circuit in Fig. 111 is equal when the loads are taken into account.

Example 3.—Compare the attenuation constants and characteristic impedances before and after loading.

(*Note*, although a double wire diagram is given in the question to give a clearer idea of the actual working conditions, the problem can be solved more easily as a single-wire T as shown in the following working.)

Example 1.—From Table II., Chap. VIII., the line angle of the unloaded circuit is

$$\theta = (0{\cdot}05802 + j\,0{\cdot}07046)2{\cdot}5 = 0{\cdot}14505 + j\,0{\cdot}17615$$

and $$Z_0 = 280{\cdot}8 \angle 39^\circ\,17'$$

First construct the equivalent T diagram of this unloaded circuit (see Ex. 5, Chap. VI.). We have

$$\tanh\frac{\theta}{2} = \tanh(0{\cdot}0725 + j\,0{\cdot}0881)$$

$$= \frac{0{\cdot}0724 + j\,0{\cdot}0884}{1 + j\,0{\cdot}0724 \times 0{\cdot}0884}$$

$$= \frac{0{\cdot}11426 \angle 50^\circ\,41'}{1{\cdot}0020 \angle 3^\circ\,40'} = 0{\cdot}1140 \angle 47^\circ\,1'$$

$$\sinh\theta = \sinh 0{\cdot}14505 \cos 0{\cdot}17615 + j \cosh 0{\cdot}14505 \sin 0{\cdot}17615$$

$$= 0{\cdot}1460 \times 0{\cdot}9845 + j\,1{\cdot}0106 \times 0{\cdot}1754$$

$$= 0{\cdot}1437 + j\,0{\cdot}1773$$

$$= 0{\cdot}2281 \angle 50^\circ\,59'$$

$$\therefore Z_0 \tanh\frac{\theta}{2} = 32{\cdot}02 \angle 7^\circ\,44'$$

and $$\frac{Z_0}{\sinh\theta} = 1231 \angle 90^\circ\,16'$$

The equivalent T of the enclosed cable loop is therefore

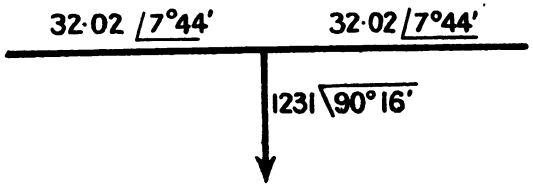

FIG. 112.

Next add the loads at each end—
(See para. 72, Chap. IV.)

$$32{\cdot}02 \angle 7^\circ\,44' = 31{\cdot}73 + j\,4{\cdot}3$$

$$3{\cdot}30 + j\,0{\cdot}066 \times 5000 = \underline{3{\cdot}30 + j\,330}$$

$$\text{Total} = 35{\cdot}03 + j\,334{\cdot}3 = 335{\cdot}7 \angle 84^\circ\,1'$$

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Hence the equivalent T of the loaded loop is

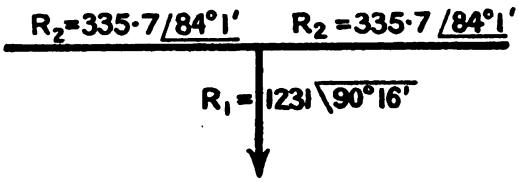

FIG. 113.

Example 2.—If θ' is the line angle of this circuit, then

$$\frac{R_2}{R_1} = \cosh\theta' - 1 = \frac{335{\cdot}7\,/\,84°\,1'}{1231\,\backslash 90°\,16'} \quad \left\{\begin{array}{l}\text{from (36), Chap. III., and the preceding figure.}\end{array}\right.$$

$$= 0{\cdot}2727\,/\,174°\,17'$$

hence

$$\cosh\theta' = 1 + 0{\cdot}2727\,/\,174°\,17'$$

$$= 0{\cdot}7292\,/\,2°\,9'$$

Hence from Dr. Kennelly's charts (see para. 31) we find that $\theta' = 0{\cdot}040 + j\,0{\cdot}754$, which is the required line angle of a uniform unloaded circuit to which the loaded equivalent circuit is equal:

$$Z_0' = \sinh\theta' \times 1231\,\backslash 90°\,16' \quad \text{[See (1), Chap. IV.]}$$

$$= 0{\cdot}686\,/\,87°\,34' \times 1231\,\backslash 90°\,16'$$

$$= 846\,\backslash 2°\,42'$$

Example 3.—$\beta' = \dfrac{0{\cdot}040}{2{\cdot}5} = 0{\cdot}016$ hyps per ml.

$$\left(= \frac{\text{real part of line angle}}{\text{length}}\right)$$

Attenuation constant before loading $= 0{\cdot}058$ hyps per ml. (See Table II., Chap. VIII.)

" " after " $= 0{\cdot}016$ hyps per ml.

Ratio of attenuation constants $= \dfrac{0{\cdot}058}{0{\cdot}016} = 3{\cdot}62$

These examples show the effect of loading a 70 lbs. cable circuit, taking into account the fact that the inductance is "lumped" in the middle of a length of 2½ miles of cable. As the cable selected and the amount and distribution of the load correspond to what is frequently actual commercial practice, it is of interest to compare the results with those obtained by uniform distribution. So far as the attenuation constant is concerned, the result is exactly the same to the first three significant figures as if the same inductance as that taken in the example were uniformly distributed over the line. The attenuation constant is calculated from the usual formula ((48), Chap. VIII.), which assumes uniform distribution. If, however, a much wider spacing of the coils were taken this attenuation would be increased, and we therefore see that the spacing adopted

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leads to satisfactory results so far as the frequency of 800 cycles per second is concerned.

As regards the characteristic impedance, the concordance at this frequency is not so good, as this example gives $846 \angle 2° 42'$ ohms, whereas calculation by the usual formula ((62), Chap. VIII.) gives $904 \angle 2° 55'$. So far as the effect on speech on long lines is concerned this difference has very little effect, but it should be carefully borne in mind.

Example 4.—A 70 lbs. cable pair having the constants given in Table II., Chap. VIII., is loaded at 2½ mile points by coils having an inductance of 132 mh. The unloaded cable has a length such that $\beta l = 1$, and by problem 2, Chap. VI., its length is known to be 17·237 miles. If telephone speaking apparatus having an impedance of $425 / 45°$ ohms is joined in series with the line at both ends, find the received current at a frequency such that $2\pi f = 5000$. The loading coils have an effective resistance of 6·6 ohms.

By Table II., Chap. VIII., we have

$$R = 25{\cdot}14 \quad L = 0{\cdot}001 \quad G = 10^{-6} \quad C = 0{\cdot}065 \times 10^{-6}$$

Adding the effective resistance and inductance of the coils we obtain

$$R = 25{\cdot}14 + \frac{6{\cdot}6}{2{\cdot}5} = 25{\cdot}14 + 2{\cdot}64 = 27{\cdot}78 \text{ ohms}$$

$$L = 0{\cdot}001 + \frac{132}{2{\cdot}5} = 0{\cdot}001 + 52{\cdot}8 = 52{\cdot}801 \text{ millihenrys}$$

$$R + j\omega L = 27{\cdot}78 + j\,264 = 265{\cdot}5 / 83° 58'$$

$$G + j\omega C = 325 \times 10^{-6} / 89° 49'$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{265{\cdot}5 / 83° 58'}{325 \times 10^{-6} / 89° 49'}} = 904 \angle 2° 55'$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(265{\cdot}5 / 83° 58')(325 \times 10^{-6} / 89° 49')}$$
$$= 0{\cdot}2937 / 86° 53'$$

Now $I_2 = \dfrac{E_1}{\left(\dfrac{Z_0^2 + Za^2}{Z_0}\right) \sinh \theta + 2Za \cosh \theta}$ by (23), Chap. III.

$$\theta = (0{\cdot}2937 / 86° 53')\ 17{\cdot}237$$
$$= 5{\cdot}0625 / 86° 53' = 0{\cdot}2752 + j\,5{\cdot}0550$$

Hence $\sinh \theta = \sinh (0{\cdot}2752 + j\,5{\cdot}0550)$ (See (78), Chap. I.)
$$= \sinh 0{\cdot}2752 \cos 5{\cdot}0550 + j \cosh 0{\cdot}2752 \sin 5{\cdot}0550$$
$$= 0{\cdot}09364 - j\,0{\cdot}9776$$
$$= 0{\cdot}9822 \angle 84° 32'$$

and $\cosh \theta = \cosh 0{\cdot}2752 \cos 5{\cdot}0550 + j \sinh 0{\cdot}2752 \sin 5{\cdot}0550$ (See (79), Chap. I.)
$$= 0{\cdot}3488 - j\,0{\cdot}2624$$
$$= 0{\cdot}4365 \angle 36° 58'$$

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$$Z_0^2 = (903{\cdot}8\angle 2^\circ\,55')^2 = 816850\angle 5^\circ\,50' = 812620 - j\,83021$$
$$Za^2 = 0 + j\,180625$$
$$\therefore\ Z_0^2 + Za^2 = 812620 + j\,97604 = 818460\angle 6^\circ\,51'$$

and
$$\frac{Z_0^2 + Za^2}{Z_0} = \frac{818460\angle 6^\circ\,51'}{903{\cdot}8\angle 2^\circ\,55'} = 905{\cdot}6\angle 9^\circ\,46'$$

$$\therefore\ I_2 = \frac{E_1}{905{\cdot}6\angle 9^\circ\,46' \times 0{\cdot}9822\angle 84^\circ\,32' + 2\times 425\angle 45^\circ \times 0{\cdot}4365\angle 36^\circ\,58'}$$
$$= \frac{E_1}{889{\cdot}4\angle 74^\circ\,46' + 371{\cdot}0\angle 8^\circ\,2'}$$
$$= \frac{E_1}{233{\cdot}7 - j\,858{\cdot}2 + 367{\cdot}4 + j\,51{\cdot}85}$$
$$= \frac{E_1}{601{\cdot}1 - j\,806{\cdot}3}$$
$$= \frac{E_1}{1006\angle 53^\circ\,18'} = \frac{E_1}{1006}\angle 53^\circ\,18' \text{ amps.}$$

If $E_1 = 10$ volts the received current is very nearly 10 milliampères at an angle of 53° 18′ and this should be compared with Example 4, Chap. VI., which shows the received current on the same circuit unloaded with the same apparatus connected to it. In this case the loaded line is electrically too short to attain to the full benefits of loading. The attenuation constant by Example 3 is 0·016 and $\beta l = 0{\cdot}016 \times 17{\cdot}237 = 0{\cdot}196$, which shows that the line is electrically short (see para. 171).

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CHAPTER X

METHODS OF MEASUREMENTS OF A.C. IMPEDANCE AND CONSTANTS OF TRANSMISSION LINES, WITH EXAMPLES OF TESTS

206. Alternating Current Measurements—Introductory Remarks.—All alternating current measurements in telephony find their *raison d'être* in the supposition that the effects of the human voice as regards attenuation in lines and apparatus may be obtained by sine wave measurements at a known frequency or frequencies, and it appears from Chap. V. that this supposition is justified within certain limits.

So far as the attenuation of voltage and current is concerned the frequency adopted in testing is generally 800 cycles per second, and the testing current one milliampère; but these may be varied to meet various circumstances. In order to find the probable effects of lines or apparatus in distorting the voice, a series of measurements over a range of frequencies is taken, and this generally comprises a range of measurements between $2\pi f = 3000$ and 7000; but these limits are by no means rigid; for example, in testing loading coils a much higher frequency is used in order to ensure that the coil is capable of transmitting the higher frequencies of the voice.

Although the standard cable can be employed to determine the effective overall attenuation in telephone circuits with sufficient accuracy if the circuit is long enough (see Chapter XI.), this method cannot be employed to find the circuit constants, although these are absolutely necessary to permit of accurate telephone circuit design. By the circuit constants the effective values of resistance, leakance, capacity and inductance of telephone lines at some definite frequency are meant, and these can only be determined by alternating current measurements. Further obvious uses of such tests in telephony are the measurement of the efficiency of transformers and its variation with frequency, the impedance of condensers, relays, etc. Machine measurements are also highly desirable in making acceptance tests of loaded cables, etc., which are made by manufacturers to a given specification.

207. Types of A.C. Apparatus used in the Engineer-in-Chief's Laboratory, G.P.O.—To meet the foregoing requirements a sine wave alternator and a device for measuring alternating currents are required. In the Engineer-in-Chief's laboratory at the Post Office these requirements are provided for in two ways.

(*a*) By means of an alternator suitable for telephonic frequencies used in connection with a Wheatstone bridge.

(*b*) By means of a "Franke" machine, which consists of an

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A.C. potentiometer in combination with an A.C. generator suitable for telephonic frequencies.

As these are representative methods some details respecting them will be given.

One of the alternators in use is a Duddell alternator suitable for telephonic frequencies. The details of this machine are, however, well known. The alternator was described by Mr. Duddell in a paper read before the Physical Society, and reproduced in the *Electrician* of April 20th, 1912, to which readers are referred.

The alternator most generally used in the Post Office Research Section in connection with the A.C. Wheatstone bridge was designep by Mr. S. A. Pollock, of the Engineer-in-Chief's Dept., G.P.O., and as it contains some specially useful features, an outline of the salient points in its construction will first be given.

208. The P.O. Sine Wave Alternator.—The alternator is driven by a direct-current motor, the usual voltages of the battery, which is reserved for this purpose only, are 40 v., 60 v., 80 v., and 100 v., to give speeds furnishing frequencies up to 2000 periods per second. The only special features of the alternator which call for mention are (*a*) the rotor has 120 teeth, which permits of a relatively high frequency without running the machine at an excessive speed; (*b*) the wave form approximates to a sine wave; (*c*) the speed of revolution per minute of the alternator shaft is the same as the frequency of the alternating current in cycles per second.

The installation comprises two highly useful features:—

(1) An electrically controlled automatic speed counter. This is fully described in Mr. C. E. Hay's paper on A.C. measurements read before the Institution of P.O. Electrical Engineers in 1912, which should be consulted. It is included in the series of professional papers published by the Institute.

(2) A speed controlling device which is very effective and which will now be described.

Reed Control of Speed.—The speed of the machine is made constant and kept constant by means of a reed control. A diagram of the arrangement is shown in Fig. 114. By the combined action of the reed governor, and a revolving metallic cylinder having an insulated segment, this cylinder being attached to the armature shaft, a resistance R_2 of 14 ohms is automatically cut in and out of the circuit, for longer or shorter periods, until a steady speed is attained. If this resistance is insufficient a further sliding resistance R_3 is provided, and by this means a preliminary regulation of speed may be made. Suppose that the machine is set to run a trifle faster than the required speed in the first instance. The reed vibrates at a fixed frequency which is constant for any fixed position of the sliding weight W on the reed; the latter can, however, be moved along the bar by hand until the required speed is obtained. The regulating commutator has the same speed as the armature and varies with it, until the regulating device reduces it and the reed to synchronism. When this is attained the synchronism results in regular and rhythmical sounds being heard in a telephone receiver, which is shunted across a resistance provided for that purpose. In the absence of synchronism of the reed and armature the sound in the telephone is confused and irregular.

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The action of the revolving commutator in controlling the speed is more easily understood by supposing the regulating action to

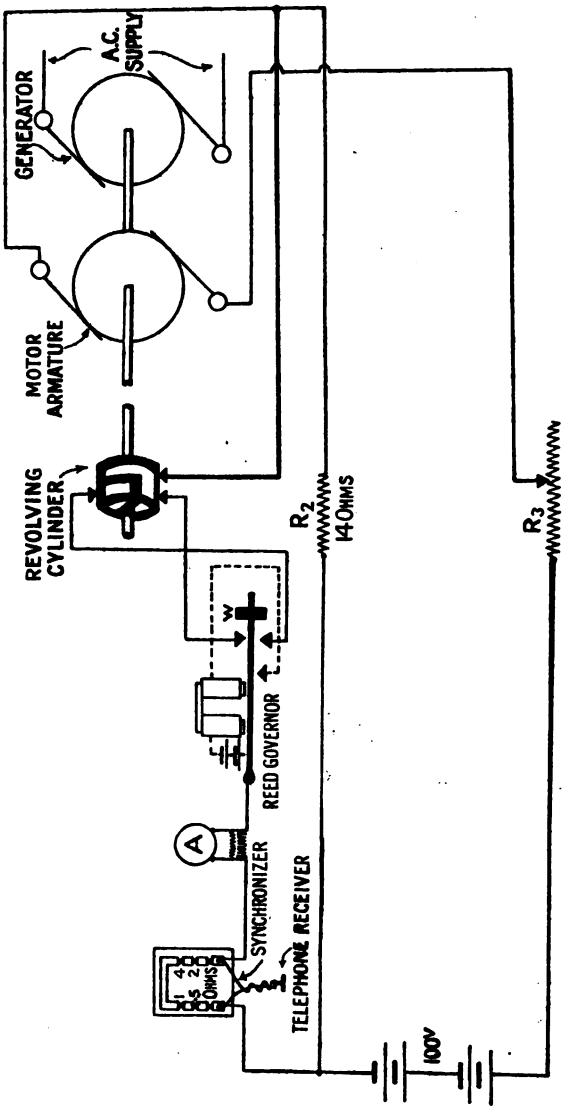

FIG. 114.—Speed Governor of Post Office Alternator for Telephonic Frequencies.

take place on a circular disc, as shown in Fig. 115, instead of on the surface of the rotating cylinder (as is the fact), as shown in Fig. 114. Referring to the former figure, one contact only of the reed is shown

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to simplify the explanation. The contact C_1 and the armature M are led to separate brushes which terminate on the rotating metal disc D. This disc has an insulated portion E marked off by a semicircle, any portion within the semicircle being insulated from the rest of the disc. The metal disc and insulated portion form, however, one uniform surface. The brush from the armature is fixed in such a position that it is not touched by the insulated portion of the disc as it rotates. The brush C, however, is by its position in metallic contact with the uninsulated disc during half its revolution, and is insulated during the remaining part of the rotation. If, therefore, the reed is on the contact C_1, at the same instant as the contact C is on the metallic portion of the disc as shown in Fig. 115, then there is a metallic circuit from the armature to the point P, and the 14 ohms' resistance is short-circuited. Suppose that this condition obtains,

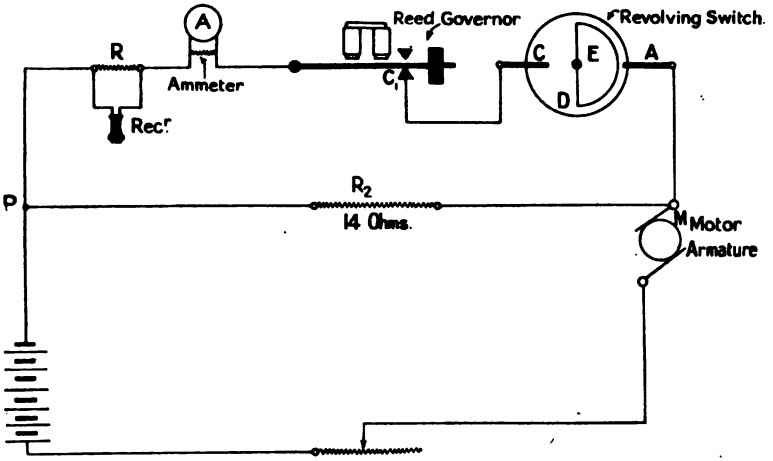

FIG. 115.—Speed Governor of Post Office Alternator. Simplified Diagram.

but that the machine is running slightly fast. In that case the armature is running more quickly then the reed, and after a few revolutions the insulated portion of the disc comes under the contact C at the same time as the reed is on the contact C_1. In that case the 14 ohms' resistance will be in the armature circuit for some portion of each revolution of D, and a reduction in speed will result. The contact of the reed with C_1 is regulated so as to continue for nearly half the time of a complete vibration of the reed, *i.e.* if the reed makes contact with its lower stop 800 times per minute, the duration of each contact is nearly $\frac{1}{1600}$ minute. If, therefore, the brush C is just entering the insulated segment E at the same instant as the reed makes contact with C_1 it will continue in contact nearly the whole of the time that C is in the insulated semicircle, if the rate of revolution of D is the same as the rate of vibration of the reed. In that case, when the reed leaves its contact, C leaves the insulated part of the disc and makes metallic contact with the

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uninsulated portion. There is, however, no circuit at this moment via the reed as it is not on C_1; and this state of things continues until C again enters E. The consequence is that the 14 ohms' resistance is nearly continuously in circuit. If, however, the contact C enters the uninsulated portion of the disc at the same moment as the reed comes to the contact C_1, then the 14 ohms' resistance is continually short-circuited.

There are a large number of intermediate positions between the two extreme positions described, and in the intermediate positions the insulated part of the disc D is in contact with the brush C at the same instant as the reed is on C_1 for some portion of the period of revolution of D, whilst for the remaining period, $ADCC_1$ and the reed are in metallic contact. These periods of insulation and short circuit may vary considerably in relative duration, and by this means a refinement of regulation is automatically achieved. If a second brush be fixed on the disc 180 degrees from C and joined to the top contact of the reed, as in Fig. 114, the regulating action described is equally divided. The deflection on the ammeter indicates by its magnitude whether the insulated disc is wholly or partly in action. The resistance of the ammeter shunt is very small.

In practice the device described is found to be very effective. An essential condition of its success is that the moving parts of the revolving armature shall not be too heavy for satisfactory regulation by the controlling device, and this condition is met in the arrangement in use at the Post Office.

209. The Alternating Current Wheatstone Bridge.—One of the arrangements used in the Post Office Engineering Research Section for measuring the impedance of telephone transmission circuits is shown by Fig. 116. (See Mr. C. E. Hay's paper on "Alternating Current Measurements," already referred to.) P is a fixed non-reactive resistance, and r and r_1 are adjustable non-reactive resistances. These resistances are of the Hay-Sullivan type and are of a high order of accuracy. C and C_1 are mica condensers of 1·999 mf. capacity, adjustable by steps of 0·001 mf. A non-reactive slide resistance and an air condenser in parallel with either C or C_1 are sometimes necessary to obtain balance, particularly when the angle of the vector impedance is very small or very large.

An earthed electrostatic screen is used between the primary and secondary windings of the transformer.

Elimination of Harmonics.—If harmonics are unduly prominent the device shown at the foot of the diagram (Fig. 116) is used in place of the receiver shown connected to the bridge. The mutual inductance M_1 and the capacity C_3 are adjusted to eliminate the third harmonic and M_2 and C_2 the fifth. This arrangement usually meets the case. The mutual inductance M and capacity C are such that $MC\omega^2 = 1$.

210. Development of Formulæ for Bridge Balance.—As in the case of the direct current Wheatstone bridge, balance is obtained when the potential fall in the arms P and A results in the potentials at the terminals of the galvanometer or receiver being equal. In the A.C. case, however, a source of alternating e.m.f. replaces the battery, and a telephone receiver the galvanometer, and the resistances in the direct current case become vector impedances. Bearing these

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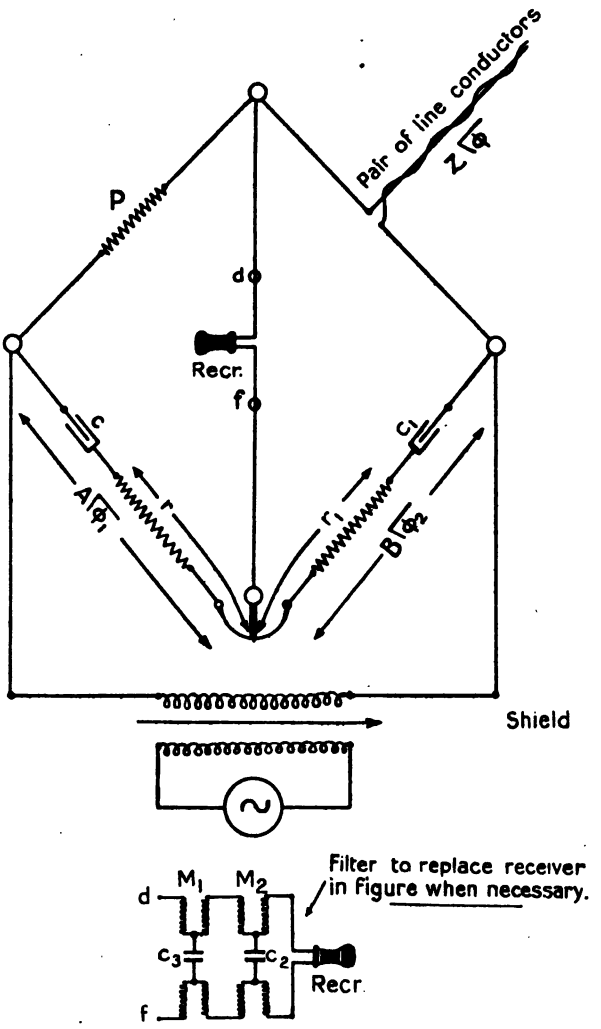

FIG. 116.—Alternating Current Wheatstone Bridge. Arranged for Measurement of Transmission Line Impedance.

points in mind we may write by analogy with the direct current case

$$\frac{P}{A\angle\phi_1} = \frac{Z\angle\phi}{B\angle\phi_2} \quad \text{and therefore} \quad Z\angle\phi = P\frac{B\angle\phi_2}{A\angle\phi_1} \quad . \quad (1)$$

The modulus $$Z = \frac{PB}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The angle $$\phi = \angle\phi_2 - \angle\phi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

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If ϕ_2 is greater than ϕ_1 then ϕ is negative; if ϕ_2 is less than ϕ_1, ϕ is positive; the former condition is assumed in Fig. 116. We have also

$$A = \sqrt{r^2 + \frac{1}{\omega^2 C^2}}$$

and

$$B = \sqrt{r_1^2 + \frac{1}{\omega^2 C_1^2}}$$

Inserting these values in (2) we obtain

$$Z = P\sqrt{\frac{r_1^2 + \frac{1}{\omega^2 C_1^2}}{r^2 + \frac{1}{\omega^2 C^2}}} \quad . \quad . \quad . \quad . \quad (4)$$

$$= P\frac{C}{C_1}\sqrt{\frac{\omega^2 C_1^2 r_1^2 + 1}{\omega^2 C^2 r^2 + 1}} \quad . \quad . \quad . \quad . \quad (5)$$

The angle ϕ_1 may be written $\tan^{-1}\frac{1}{\omega C r}$ . . . . . (6)

Similarly the angle ϕ_2 may be written $\tan^{-1}\frac{1}{\omega C_1 r_1}$ . . . (7)

Whence $\phi = \sqrt{\tan^{-1}\frac{1}{\omega C_1 r_1}} - \sqrt{\tan^{-1}\frac{1}{\omega C r}}$ by (3) . . . . (8)

211. Details of an Impedance Measurement.—If the vector impedance of a uniform transmission line as measured from the sending end, (*a*) when the distant end of the circuit is open, and (*b*) when it is closed, are known, the attenuation constant, wave-length constant, and the resistance, leakance, inductance and capacity of the circuit, can be calculated from the measured impedances. An example of this is fully worked out after the details of the Franke machine. What is here required, then, is the measurement of two impedances, and this can be done by the Wheatstone bridge with the aid of the formulæ already developed.

The following are the details of the actual measurement of a sending end impedance of a continuously loaded cable with the distant end closed.

When balance in the bridge was obtained the various arms had the following values, the frequency being such that $2\pi f = 10{,}000$:—

$P = 100$ ohms
$r = 370$,,
$C = 0{\cdot}55$ mfd.
$r_1 = 1000$ ohms
$C_1 = 0{\cdot}020$ mfd.

By (5) we have

$$Z = 100 \times \frac{0{\cdot}55}{0{\cdot}02} \times \sqrt{\frac{(10^4 \times 0{\cdot}020 \times 10^{-6} \times 1000)^2 + 1}{(10^4 \times 0{\cdot}55 \times 10^{-6} \times 370)^2 + 1}}$$

$$= \frac{100 \times 0{\cdot}55}{0{\cdot}02}\sqrt{\frac{(0{\cdot}2)^2 + 1}{(2{\cdot}035)^2 + 1}}$$

$$= 2750\sqrt{\frac{1{\cdot}04}{5{\cdot}14}} = 2750 \times 0{\cdot}45$$

$$= \underline{1238}$$

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The corresponding angles are by (6) and (7) from the preceding calculations

$$\phi_1 = \tan^{-1}\frac{1}{2{\cdot}035} = \tan^{-1} 0{\cdot}491 = 26^\circ\ 9'\ 6''$$

$$\phi_2 = \tan^{-1}\frac{1}{0{\cdot}2} = \tan^{-1} 5{\cdot}0 = 78^\circ\ 41'\ 24''$$

From (8) $\phi = 78^\circ\ 41' - 26^\circ\ 9' = 52^\circ\ 32'\ 18''$.

As ϕ_2 is greater than ϕ_1 the angle is negative. The measured impedance is therefore

$$Z_0 \angle \phi = 1238 \angle 52^\circ\ 32'$$

Similarly the impedance with the distant end open could be measured.

In this case the angles have been found from suitable tables from the known tangents, deduced from the measurements. If, however, great accuracy is required, particularly if the angles are very large, the greatest accuracy is obtained by finding the angle by trigonometrical formulæ. For example, the angle in the preceding case might have been found from the known trigonometrical relation

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}$$

In the preceding example this becomes

$$\phi = \angle\tan^{-1}\frac{1}{\omega C_1 r_1} - \angle\tan^{-1}\frac{1}{\omega C r} = \angle\tan^{-1}\frac{\frac{1}{\omega C_1 r_1} - \frac{1}{\omega C r}}{1 + \frac{1}{\omega^2 C C_1 r r_1}} \quad (9)$$

$$= \angle\tan^{-1}\frac{5{\cdot}0 - 0{\cdot}491}{(5 \times 0{\cdot}491) + 1} = \angle\tan^{-1}\frac{4{\cdot}509}{3{\cdot}455} = \angle\tan^{-1} 1{\cdot}305$$

$$= \angle 52^\circ\ 32'\ 16''$$

This only differs from the value previously found by 2″ and does not materially affect the result. In fact, the bridge readings would be liable to a greater variation.

Other more complex angular combinations may be resolved in a similar way.

The special advantage of the Wheatstone bridge is that it is capable of measuring impedances with great precision in the hands of skilled experimentalists, especially if a sine-wave alternator with an efficient speed governor is available, as in this case.

A number of other developments of formulæ in connection with impedance measurements by means of the apparatus described will be found in the paper by Mr. Hay already referred to.

212. The Drysdale D.C.—A.C. Potentiometer.—This potentiometer was invented by Dr. C. V. Drysdale for the purpose of measuring alternating potential differences on the potentiometer method, *i.e.* by balancing any sine wave voltage to be measured against another

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voltage derived from the same source, and supplied by the potentiometer, which is arranged to modify the measuring voltage to give any required modulus and angle.

The apparatus as made and supplied by Mr. H. Tinsley is adapted for measuring either direct or alternating current potential differences. The A.C. apparatus is well adapted for measuring the range of voltage covered by telephonic transmission, say a fraction of a

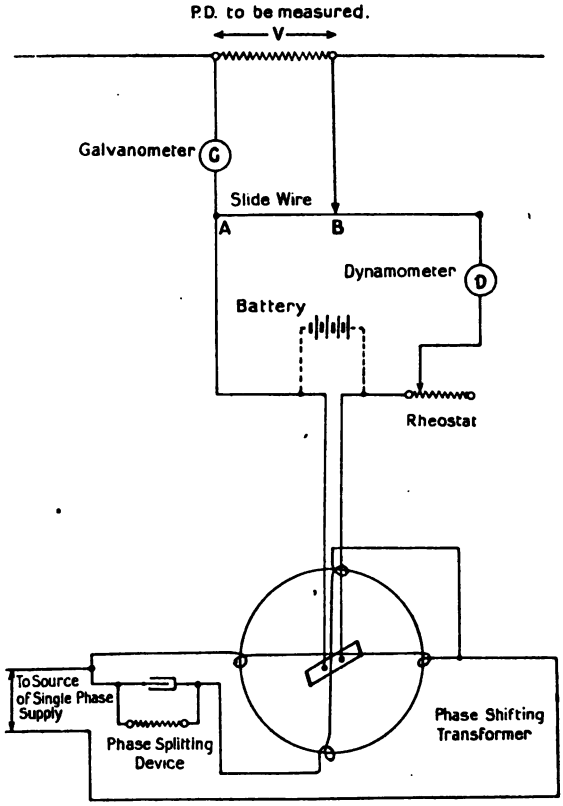

FIG. 117.—Drysdale A.C.—D.C. Potentiometer. Explanatory Diagram.

volt to 10 volts. The principle of its operation may be readily understood by recalling the principle of the D.C. potentiometer. Fig. 117 is a simplified diagram illustrating the A.C. case, but if the phase-shifting device is replaced by a battery as shown by the dotted lines, and the vibration galvanometer is substituted by a D.C. galvanometer, we have the direct current apparatus. If, then, a current I passes through the known resistance R and the voltage at its terminals is $RI = V$, this voltage, which is to be measured, will cause a current

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to flow through G and the slide wire. If a current from the battery replacing the phase shifter be sent round the galvanometer in the opposite direction, and if the slide wire be regulated until no current passes through G, then the voltage at the terminals of AB is equal to that at the terminals of R, and this may be found from the potentiometer readings. Usually the current in the slide wire is regulated by the rheostat to have a value convenient for the ready reading of the volts from the indications of the slide wire. The dynamometer permits of the current in circuit being readily known. In the A.C. case the voltage across R is a vector quantity. The function of the slide wire is then to regulate the amount or modulus of the opposing voltage, whilst the phase-shifting device permits of any phase being associated with the modulus.

The Vibration Galvanometer is constructed so as to give vibrations of a maximum amplitude on the resonance principle, when a current of some definite frequency is passed through it. Such a galvanometer gives a sensitive condition for balance, because it is arranged to give a maximum deflection at the required frequency, and it also indicates (when the maximum deflection is observed) that the correct frequency is attained. For the range of frequencies required in telephony a Duddell vibration galvanometer is a convenient instrument. By varying the length of the vibrating wire carrying the current, which is suspended in a strong magnetic field, it may be tuned to frequencies between 100 and 1800 cycles per second. (See "On a Bifilar Vibration Galvanometer," W. Duddell, *Proc.* Physical Society, February, 1910.) A telephone receiver may be used for balancing purposes in place of the galvanometer if the frequency of the test is suitable.

The Phase-shifting Device consists of a phase-shifting transformer with two primary windings. Four coils are wound and placed 90 degrees apart round a circle, and the diametrically opposite coils are connected to make two pairs. If these two pairs are connected to a two-phase source of supply differing 90 degrees in phase, a rotating magnetic field is produced at the centre of the circle. If the secondary of the transformer is wound with wires in one plane and placed in position at the centre of the circle for maximum cutting of lines due to one pair of primary coils, the effect due to the other pair will be nil. If now the coil is rotated 90 degrees, the coils which produced a minimum effect will now produce a maximum effect and vice versa, but the phase will be 90 degrees different. If the secondary coil be placed in an intermediate position between the two extremes named, the phase will be modified in proportion to the degree of rotation. This illustrates the principle of the phase-shifting transformer. By this means an alternating current from the same source as the A.C. voltage to be measured may be changed in phase as may be required. In practice it is frequently necessary to work from a single phase supply, and the device known as "*phase splitting*" is then resorted to. An idea of the principle underlying this apparatus may be obtained by remembering that if an alternating current is passed through a resistance and capacity in parallel, the current passing through the resistance will be in phase with the voltage, but the portion of the current passing through the condenser will lead by 90 degrees. If the reactance

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of the condenser is equal to the resistance which is in phase with the voltage, two currents of equal magnitude but differing by 90 degrees are obtained. This ideal arrangement does not hold as the coils of the transformer are in circuit, and a circular revolving field would not be given because the transformer has both resistance and inductance, which alter the phase relations. In practice the electrical values are suitably adjusted by the manufacturer, and can be regulated according to the frequency of the supply.

The phase-shifting transformer arranged for connection to a single phase source of supply is shown theoretically by Fig. 117.

Fig. 118 shows a more detailed theoretical diagram, whilst the actual arrangement of the potentiometer is shown by Fig. 119. These two latter diagrams are given by Dr. Drysdale in an article

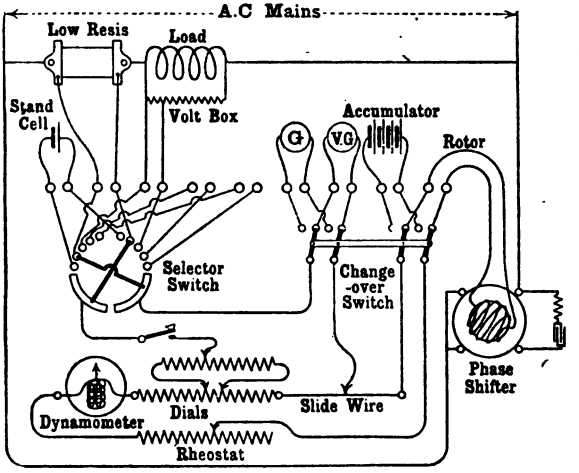

FIG. 118.—Drysdale A.C.—D.C. Potentiometer. Skeleton Diagram.

on "The Use of the Alternate-Current Potentiometer for Measurements on Telegraph and Telephone Circuits," in the *Electrician* of August 1st, 1913.

It will be seen that by means of the potentiometer an alternating current is supplied via the secondary of the phase-shifting transformer, which can be varied in magnitude and phase, and made to balance the voltage to be measured. The angle is read by the indication of the phase-shifter dials and the magnitude by that of the potentiometer calibration.

The voltage and current can both be determined, as the resistance R is known. We then have $\frac{V}{I} = Z$, which gives the impedance. If the vector impedances of a uniform transmission circuit are measured from the sending end of the line (*a*) with the distant end of the circuit open, and (*b*) with the distant end closed, the attenuation constant and the other circuit constants can, as already mentioned,

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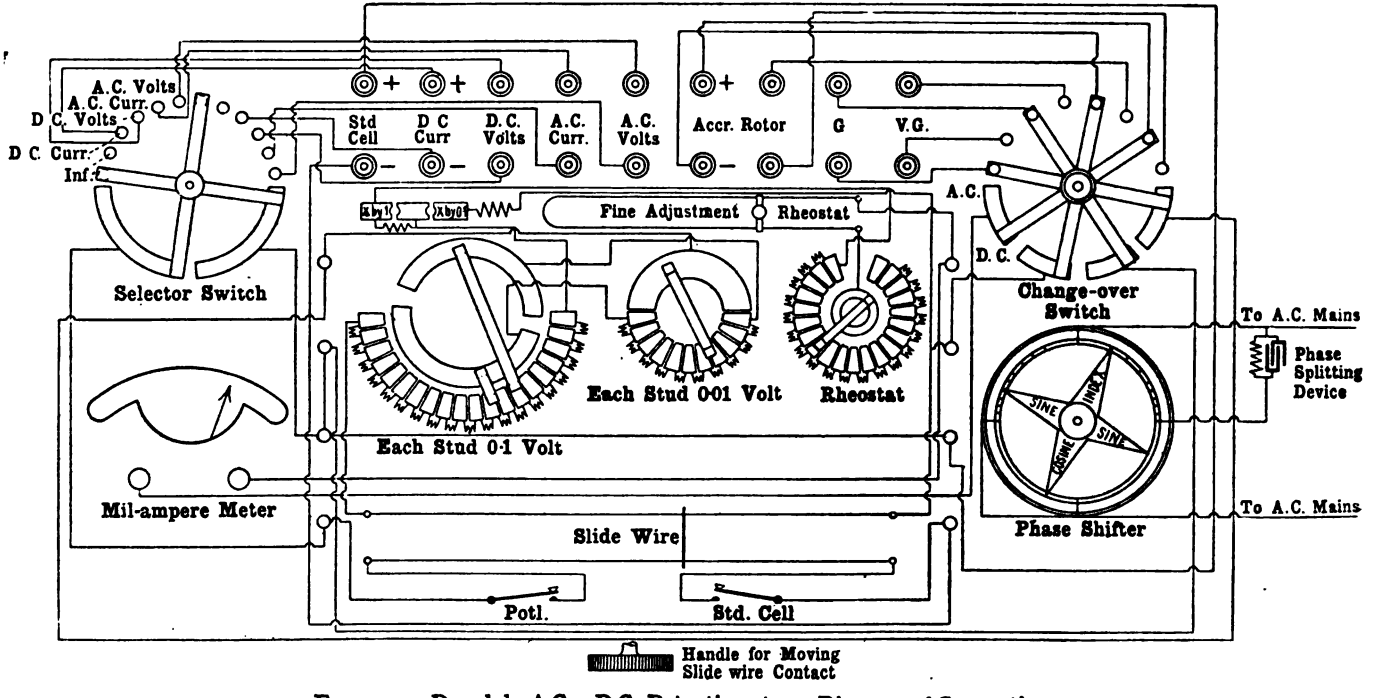

FIG. 119.—Drysdale A.C.—D.C. Potentiometer. Diagram of Connections.

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be deduced by the methods shown in this chapter after the description of the Franke machine.

213. The Franke Machine.—This machine was invented by Dr. A. Franke, of Berlin, and a detailed description was given in the *Elektrotechnische Zeitschrift* in 1891 (page 448).

Since that time the construction of the machine has been materially improved, the principal change being that the most modern form runs on a vertical instead of a horizontal axis.

Description of the Machine.—The machine is designed to supply alternating current varying from a fraction of a milliampère to possibly 10 milliampères, which fairly covers the range of voice currents at the sending end of telephonic circuits at present. The range of frequency generally used is from $2\pi f = 3000$ to $2\pi f = 7000$, but the machine can be run up to a frequency of about 2000 cycles per second. It is designed to measure the amplitude and phase of the voltage and current at the terminals of the impedance under test on the potentiometer principle, and is particularly suited for obtaining commercial results with ease and comparative rapidity, but its accuracy of angle measurement is limited, largely owing to speed variation.

The alternating current is generated by the motion of a motor-driven magnetic field, rotating round two non-revolving circular armatures. The revolving field induces currents of approximately sine wave form in the armatures, and thus provides an A.C. supply. The speed of the motor can be varied by voltage variation and by means of resistances in its curcuit. Owing to the weight of the rotor automatic control of the speed has not yet been attained. One end of the revolving device which produces the alternating currents is shown in Fig. 120. Here we have two wrought-iron concentric discs with 40 rounded teeth arranged as shown, the summits of the teeth being opposite to each other. One of the cylindrical armatures referred to lies in the circular space between the two sets of teeth as shown. The two circular discs enclosing it are carried on the same vertical spindle and revolve around it. The circular discs form one end of a wrought-iron cylindrical box which carries a magnetizing coil, and at the other end of the cylinders the arrangement of discs with their teeth and armature is repeated. The two armatures are, however, quite independent of each other. A sectional sketch showing the internal arrangement is given in Fig. 121. The rotating cylinders, which are fitted to the spindle by brass rings, are indicated by the letters C, C_1, the circular discs by F, F_1, F_2, F_3, and the non-rotating armatures by TA and BA. Other details are given at the foot of the figure.

Current is led to the magnetizing coil D from the slip rings, and a steady magnetic field passes across the teeth of the circular discs shown in Fig. 120. These as stated are 40 in number on each disc, and when the machine rotates a variable field is superimposed on the steady one. It is, of course, the variable field which produces the alternating current, the frequency being $40n$, where n is the number of revolutions of the discs per second. Each of the armatures consists of an ebonite base with 80 slots cut horizontally and vertically: the form of the winding is shown by Fig. 120. The shape of the teeth is so designed that when the pole pieces revolve the lines

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of force projected through the armature windings vary in such a way as to set up an electromotive force of approximately sine wave form in the armatures. The voltages induced in the horizontal part of the windings cancel each other. The slip rings G, Fig. 121, include provision for a current supply for the motor, and a commutator is fixed upon the rings to provide for a supply of the armature

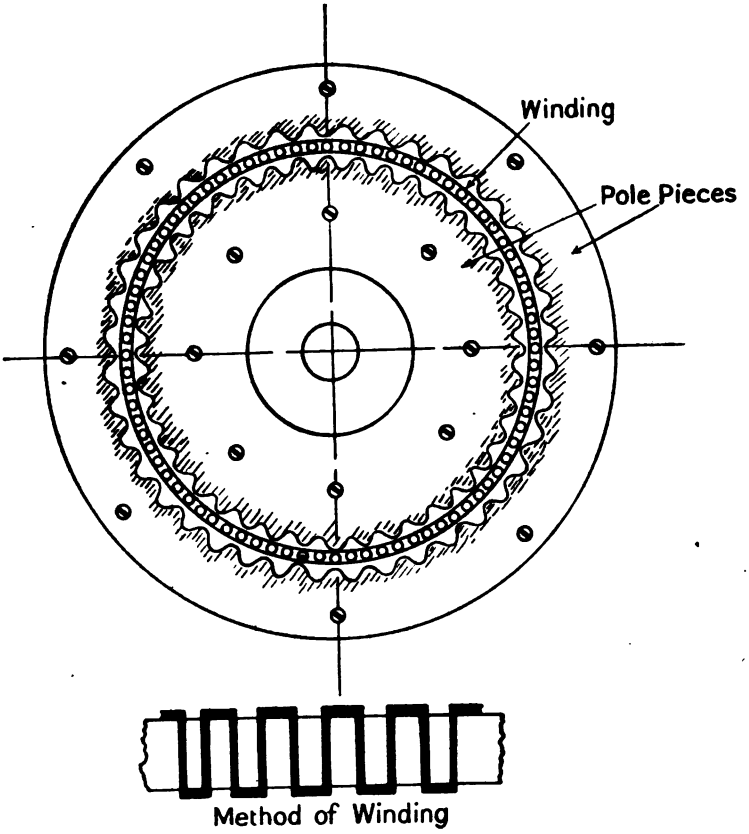

FIG. 120.—Franke Machine. Revolving Pole Pieces.

current to a frequency-measuring device, consisting of reeds which vibrate at their own natural frequency. When the amplitude of their motion is sufficient they indicate the rate of vibration, which is marked upon an indicator conveniently placed for the purpose. The machine was made by Messrs. Siemens and Halske.

Movement of the Armatures.—In order that the voltages in the two armatures may be balanced against each other, the upper armature is made to move vertically by means of a regulating screw

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not shown in the figure. By this means the armature may be raised from the exciting field above the teeth. The amount of elevation is indicated by a graduated scale on the outer cylinder surrounding the armature.

To permit of the armatures being brought into electrical phase with each other the bottom armature is so constructed that it can be rotated in a horizontal sense. The number of degrees rotated through is indicated by a graduated scale fixed above the wheel which produces the horizontal motion. Thirty-six turns of the phase-shifting wheel are required to produce a revolution of the lower armature through 360 electrical degrees.

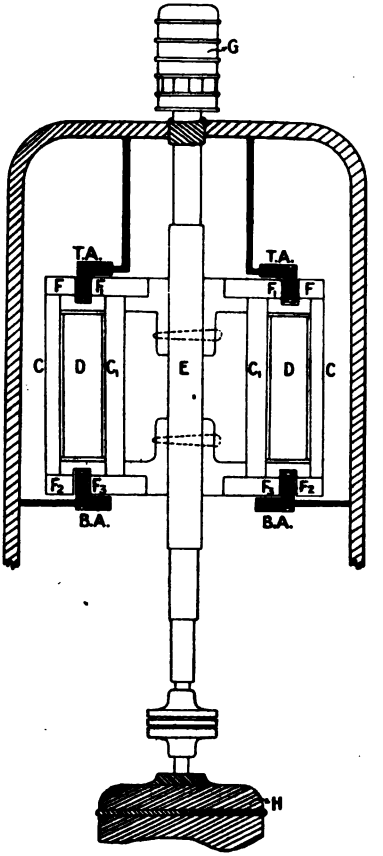

FIG. 121.—Franke Machine. Sectional Sketch.
T.A. = Top armature.
B.A. = Bottom armature.
C = Outer wrought-iron cylinder.
C_1 = Inner wrought-iron cylinder.
D = Magnetizing coil.
E = Spindle.
F, F_1, F_2, F_3 = Iron pole pieces.
G = Slip rings for magnetizing current supply and commutator in connection with frequency-measuring device.
H = Motor.

214. Methods of using the Franke Machine. — *Impedance Test.*—Fig. 122 shows a common arrangement of the Franke machine for this test. The lower armature is permanently connected as shown. By means of a switch fixed on the machine the telephone and upper armature are first joined to the position marked 1 and 2 as indicated by the solid lines in the diagram; the upper armature and telephone can, however, be moved to the position marked 2 and 3 by means of the switch, and the connections in this case are shown by dotted lines. The theoretical arrangement for position 1–2 is shown by Fig. 123A, and that for position 2–3 by Fig. 123B. In order to test in position 1–2, suppose that the upper armature, to begin with, is in its lowest position, where it gives the maximum voltage. As the armature is moved upward by means of its regulating screw the voltage becomes gradually reduced owing to the fact that the armature is withdrawn from the influence of the magnetizing coil. Every

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position of the armature as indicated by its graduated scale is associated with a definite voltage relation to the full voltage, the exact relation is determined by calibration and recorded on a permanent curve. The phase-controlling device on the bottom armature should indicate zero at this stage. This does not mean that there is no electrical phase difference between the top and bottom armatures, but that the phase of the bottom armature is taken as the point of origin for comparative purposes.

Details of Tests.

Let the e.m.f. at the terminals of the upper armature and impedance Z in position 2–3 be E.

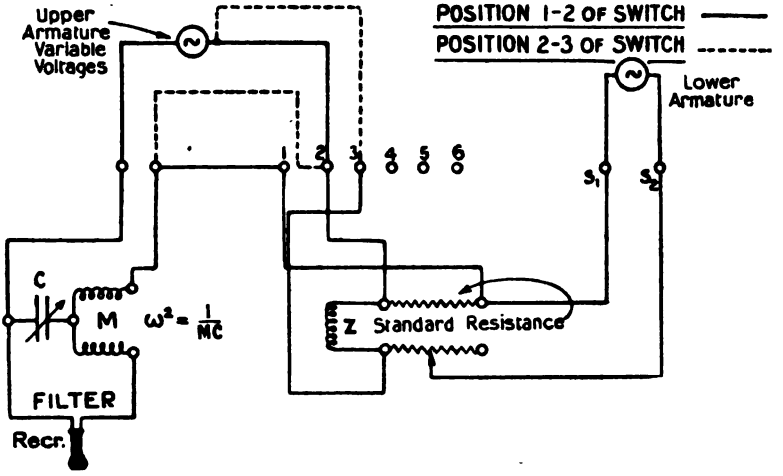

FIG. 122.—Franke Machine. Connections for Impedance Test.

Let the current in the standard resistance and in Z (pos. 1–2) be I.

Then $$Z = \frac{E}{I}$$

Method I.—Position 1–2 of the Switch. To obtain an Expression for the Current.—Let the apparatus impedance to be tested be Z.

First raise the upper armature until the fundamental note in the receiver is reduced as much as possible. The lower armature is next rotated until the sound in the receiver is further reduced as far as practicable. If the machine fundamental is not reduced to silence the upper armature must again be regulated, and afterwards the lower one, and so on until silence is obtained. It is probable that some harmonics, particularly the third, will be heard, even when the fundamental is silenced, but in practice it is possible largely to neglect them in regulating the machine, so as to obtain a correct balance for the fundamental. If necessary a filter may be used to eliminate the harmonics, as shown in Fig. 122. The upper and lower armatures tend to send currents in opposite directions through the receiver, but if silence in the receiver is obtained, the

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voltage in the upper armature is equal in magnitude and phase to the voltage across the standard resistance (see Fig. 123A). Next note the position of the upper armature and find the corresponding voltage E_1 from the calibration curve, and note the angle through which the lower armature has been turned. These two components give the voltage across the standard resistance, say $E_1 / \underline{\phi_1 + \omega t} = E_1 e^{j(\phi_1 + \omega t)}$.

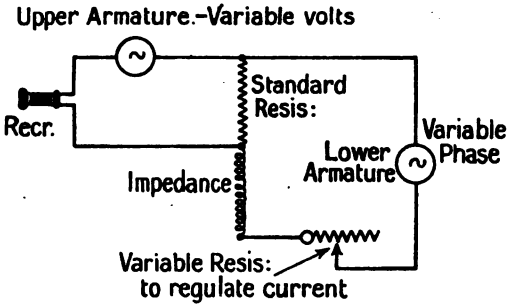

FIG. 123A.—Franke Machine. Impedance Test. Position 1 and 2.

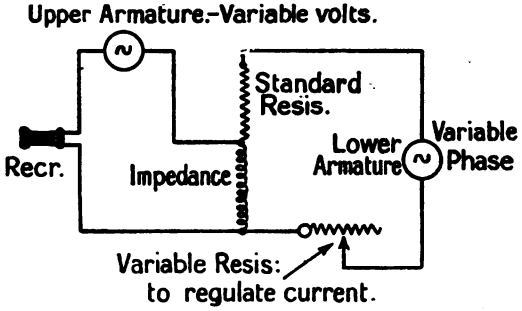

FIG. 123B.—Franke Machine. Impedance Test. Position 2 and 3.

If I is the current flowing through the standard resistance R then

$$IR = E_1 e^{j(\phi_1 + \omega t)}$$

whence

$$I = \frac{E_1}{R} e^{j(\phi_1 + \omega t)} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Position 2-3 of the Switch. To determine the Voltage.—Balance the volts and phase as before. The voltage across the impedance is the same as that at the terminals of the upper armature, as shown by Fig. 123B. Find the magnitude of the volts E_2 from the position of the upper armature and calibration curve, combined with the phase rotation $\phi_2 + \omega t$ of the lower armature. The voltage is then

$$E = E_2 e^{j(\phi_2 + \omega t)} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

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Dividing (11) by (10) we obtain

$$Z = \frac{E}{I} = \frac{E_2 e^{j(\phi_2 + \omega t)}}{\frac{E_1}{R} e^{j(\phi_1 + \omega t)}} = \frac{E_2 R}{E_1} e^{j(\phi_2 - \phi_1)} \quad . \quad . \quad (12)$$

This method requires four readings and a calibration of the machine.

Method II.—Phase and Resistance only changed in one of the Tests.— This is shown by Fig. 124. Here E_2 is kept constant and R

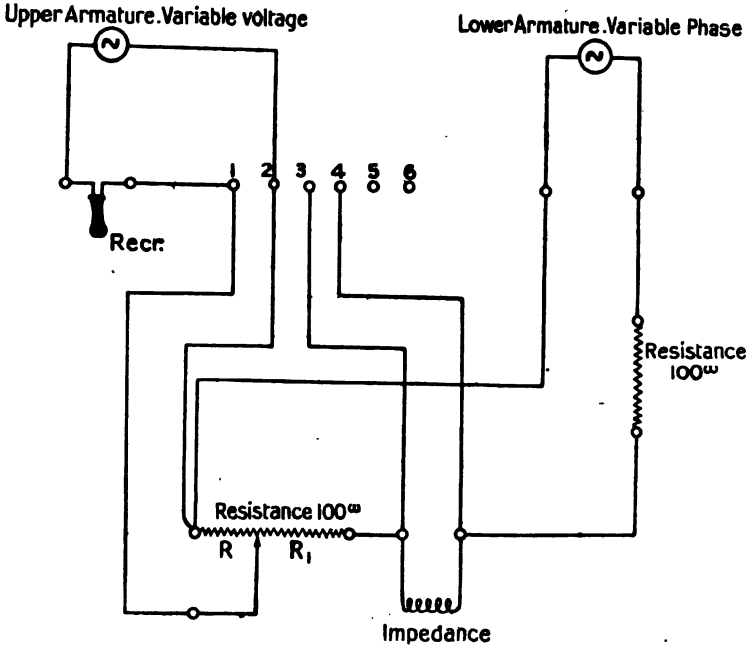

FIG. 124.—Franke Machine. Impedance Test. Arrangement for Constant Current.

varied. In order that the current in Z may not vary it is necessary to make use of a regulating resistance, as shown in the figure. These resistances need not be alike, but the sum of the two must be constant.

The Test.—Place the switch in position 3–4 and proceed as before. The voltage is then

$$E = E_2 e^{j(\phi_2 + \omega t)} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and this is the voltage across the impedance.

Position 1–2 *of the Switch.*—The upper armature and receiver are now on 1 and 2 (Fig. 124). Leave the upper armature unaltered,

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but vary the sliding resistance and phase until a balance is obtained The resistance in series with the upper armature has the same voltage at its terminals as was obtained in position 3–4, and the current in R is the same as the current in Z. We then have

$$RI = E_2 e^{j(\phi_1 + \omega t)}$$

whence

$$I = \frac{E_2}{R} e^{j(\phi_1 + \omega t)} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Dividing (13) by (14) we have

$$Z = \frac{E}{I} = R \frac{E_2 e^{j(\phi_2 + \omega t)}}{E_2 e^{j(\phi_1 + \omega t)}} = R e^{j(\phi_2 - \phi_1)}$$

It may be necessary in some cases to have a much larger resistance than the 100 ohms shown in Fig. 124.

In this case we have only three readings and no calibration of the upper armature is necessary.

Use of Balancing Resistance.—A further refinement in balancing is sometimes made by providing an extra balancing resistance which is thrown into circuit on one side or the other of the sliding resistance, when an approximate balance has been obtained. If this resistance is variable and is switched rapidly to one side of the sliding resistance and then to the other side, the rapid comparison permits of observing on which side the greater noise is observed, and it is then known that this side is more out of balance than the other. When the noise is equal on both sides balance is obtained.

215. Details of Actual Test in accordance with Figs. 122 and 123. **Test of an Aerial Double Wire Telephone Circuit consisting of Conductors weighing 800 lbs. per Mile.**—Length 29·5 miles. Standard resistance used 1000 ohms. Frequency $2\pi f = 5000$.

When the connections were arranged as in Fig. 123, position 2–3 of switch, with the distant end open, the reading obtained from the machine both as regards modulus and phase is shown in the following table in col. 1, and the corresponding reading on the calibration curve is given in col. 3. With the switch in position 1–2, and with the distant end still open, the reading on the machine is given in col. 2 and the corresponding calibration figures in col. 4. The impedance with the distant end open is then by (12)

$$Z = \frac{E_2}{E_1} R e^{j(\phi_2 - \phi_1)}$$

$$= \frac{\cdot 233}{\cdot 455} \times 1000 e^{j(78^\circ 25' - 157^\circ 45')} = 512 e^{-j79^\circ 20'} = 512 \angle -79^\circ 20'$$

In order to find the impedance measured at the sending end with the distant end closed, the same series of readings is taken as those taken for the test with the ends open; the results of the closed circuit impedance test are given in the bottom line of the table. The impedance in this case is then calculated from (12) as before.

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TABLE I.

FRANKE MACHINE READINGS.

ω	Readings.				Ratios.				Impedance.	Terminal conditions.
	1 Volts.		2 Current.		3 Volts.		4 Current.			
	M	Angle.	M	Angle.	M	Angle.	M	Angle.		
5000	73·0	78° 25′	108·9	157° 45′	·233	78° 25′	·455	157° 45′	$512\angle 79° 20'$	Open.
,,	95·0	203° 27′	108·8	141° 50′	·370	203° 27′	·454	141° 50′	$815\angle -61° 37'$	Closed.

In some cases a small calibration correction of the angle reading is necessary, but the correction never affects the reading to a greater extent than a few minutes of arc.

The following table gives the results of a series of similar tests on the same circuit at different frequencies, including the circuit constants calculated from the open and closed impedances:—

TABLE II.

SUMMARY OF RESULTS OF TESTS OF 300 LBS. AERIAL COPPER WIRE.

Length of circuit under test = 29·5 miles. Atmospheric conditions—Damp.

$\omega = 2\pi f$	Impedances measured by Franke machine.		Constants per mile loop deduced from the measured impedances.					
	Z ohms. Circuit ends open.	Z_1 ohms. Circuit ends closed.	Z_0 ohms.	R ohms.	L mhys.	G 10^{-6} mhos.	C mfds.	β hyps.
3000	$1064\angle 83° 56'$	$430\angle -55° 41'$	$677\angle 14° 7'$	6·66	3·78	1·09	·0096	·00549
4000	$724\angle 82° 7'$	$594\angle -60° 10'$	$655\angle 10° 59'$	6·75	3·82	1·27	·0098	·00569
5000	$512\angle 79° 20'$	$815\angle -61° 37'$	$646\angle 8° 52'$	6·73	3·81	1·45	·0097	·00574
6000	$358\angle 74° 51'$	$1158\angle -59° 33'$	$644\angle 7° 39'$	6·87	3·82	1·38	·0097	·00582
7000	$240\angle 66° 8'$	$1732\angle -53° 5'$	$645\angle 6° 32'$	6·84	3·82	1·49	·0094	·00583

To illustrate the method of calculating the constants the following details are given of the calculations for $\omega = 5000$ from Table II. The same method of calculation of the constants is used in connection with the A.C. bridge readings, or any others which give the open and closed sending and impedance.

By (5), Chap. III., $Z = Z_0 \coth \theta = Z_0\left(\frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}\right) = 512\angle 79° 20'$.

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By (6), Chap. III., $Z_1 = Z_0 \tanh\theta = Z_0\left(\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right) = 815\,\underline{/\,61^\circ\,37'}$,

By (7), Chap. III., $Z_0 = \sqrt{ZZ_1} = \sqrt{512\,\overline{\backslash 79^\circ\,20'} \times 815\,\underline{/\,61^\circ\,37'}}$

$= 646\,\overline{\backslash 8^\circ\,52'}$

By (5) and (6) we have

$$\sqrt{\frac{Z_1}{Z}} = \sqrt{\frac{\sinh^2\theta}{\cosh^2\theta}} = \tanh\theta = \text{(say) } h = \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} = \sqrt{\frac{815\,\underline{/\,61^\circ\,37'}}{512\,\overline{\backslash 79^\circ\,20'}}}$$

$$= 1{\cdot}262\,\underline{/\,70^\circ\,28'}$$

By (18), Chap. I., this may be written

$$h = A\,\underline{/\,\phi} = a + jb = 0{\cdot}4216 + j\,1{\cdot}188$$

Now $e^{2\theta} = e^{2\gamma l} = \frac{1+h}{1-h}$

Proof.—

$$\left.\begin{aligned} &\frac{e^{\theta}}{e^{\theta}} \times \frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}} = \tanh\theta = \frac{e^{2\theta}-1}{e^{2\theta}+1} \\ &\therefore\ e^{2\theta} = \frac{1+\tanh\theta}{1-\tanh\theta} = \frac{1+h}{1-h} \end{aligned}\right\} \quad (15)$$

for $$\frac{1+\tanh\theta}{1-\tanh\theta} = \frac{1+\dfrac{e^{2\theta}-1}{e^{2\theta}+1}}{1-\dfrac{e^{2\theta}-1}{e^{2\theta}+1}} = \frac{e^{2\theta}+1+e^{2\theta}-1}{e^{2\theta}+1-e^{2\theta}+1} = e^{2\theta}$$

We therefore have

$$e^{2\theta} = \frac{1+h}{1-h} = \frac{1+(a+jb)}{1-(a+jb)} = \frac{(1+a)+jb}{(1-a)-jb} = \frac{1{\cdot}4216 + j\,1{\cdot}188}{0{\cdot}5784 - j\,1{\cdot}188}$$

$$= \frac{1{\cdot}853\,\underline{/\,39^\circ\,53'}}{1{\cdot}320\,\overline{\backslash 64^\circ\,3'}} = 1{\cdot}403\,\underline{/\,103^\circ\,56'}$$

From this value of $e^{2\theta} = \frac{1+h}{1-h} = \frac{1+\tanh\theta}{1-\tanh\theta}$ the value of $\theta = \gamma l$ is found as follows:—

$$\theta = \gamma l = \tfrac{1}{2}\log_e \frac{1+h}{1-h} = \tfrac{1}{2}\log_e \frac{1+\tanh\theta}{1-\tanh\theta} = \tfrac{1}{2}\log_e 1{\cdot}403\,\underline{/\,103^\circ\,56'}$$

The attenuation length βl is half the $\log_e$ of the modulus, or $\frac{1}{2}\log_e 1{\cdot}403 = \frac{0{\cdot}3386}{2}$, and since in this case $l = 29{\cdot}5$ miles, $\beta = \frac{0{\cdot}3386}{2\times 29{\cdot}5}$ $= 0{\cdot}00574$.

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The wave length constant α is obtained thus [see (27), Chap. I.]

$$2\alpha l = 103° 56' = 1{\cdot}8140 \text{ radians} \quad \therefore \alpha = \frac{1{\cdot}8140}{59} = 0{\cdot}0307$$

To deduce the Circuit Constants

$$\theta = \gamma l = l(\beta + j\alpha) = \frac{0{\cdot}3386 + j\,1{\cdot}8140}{2} = \frac{1{\cdot}846\,\underline{/79° 25'}}{2}$$

Now by (7), Chap. II., $\gamma \times R_0 = R$, and in the alternating current case this becomes $\gamma \times Z_0 = (R + j\omega L)$. We therefore have

$$R + j\omega L = \gamma \times Z_0 = \frac{1{\cdot}846\,\underline{/79° 25'}}{2 \times 29{\cdot}5} \times 646\underline{\backslash 8° 52'} = 20{\cdot}2\,\underline{/70° 33'}$$

$$= 6{\cdot}726 + j\,19{\cdot}05$$

Hence $R = 6{\cdot}726$ ohms per mile loop

and $L = 3{\cdot}81$ millihenrys per mile loop.

Similarly by (8), Chap. II., $\frac{\gamma}{R_0} = G$, and in the A.C. case $\frac{\gamma}{Z_0} = G + j\omega C$.

Whence $\frac{\gamma}{Z_0} = G + j\omega C = \frac{1{\cdot}846\,\underline{/79° 25'}}{2 \times 29{\cdot}5 \times 646\underline{\backslash 8° 52'}} = 0{\cdot}0000484\,\underline{/88° 17'}$

$$= (1{\cdot}45 + j\,48{\cdot}4)10^{-6}$$

Hence $G = 1{\cdot}45$ micro-mhos per mile loop

and $C = 0{\cdot}0097$ mfds.

Conclusions from the Tests on 300 *lbs. Aerial Circuit.*—It will be seen from Table II. that the variation of the electrical constants with frequency between the limits taken is relatively small, and this remark also applies to the variation of the attenuation constant. These values should be compared with the calculated ones for the same circuit in Table II., Chap. VIII., when it will be seen that there is some difference between the calculated and observed results. The difference is very largely accounted for by the fact that the 300 lbs. circuit contains relatively short lengths of lesser gauge and some small lengths of underground cable. The leakance is also above the normal on account of the damp weather. As regards the inductance and capacity, however, it is probable that the large number of wires on the same poles affect the result to some extent. The capacity was tested in addition by a discharge method and found to agree with the result given in the preceding table.

216. Additional Tests on a 300 lbs. Aerial Line.—The following tests relate to tests of 34·9 miles of 300 lbs. circuit containing no underground sections but having a length of 2 miles of 150 copper conductors in it. Three series of tests were made by means of an A.C. machine on the following circuits :—

(*a*) A twisted loop.

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(*b*) A phantom circuit (two pairs of wires revolved around each other).

(*c*) A single wire circuit.

TABLE III.

DETAILS OF TESTS.

Circuit.	$2\pi f$	Constants per mile deduced from the measured impedances.						Direct current tests.	
		Z_0	R ohms.	L mh.	G mhos.	C	β	R ohms.	Insulation.
a	5000	$639\angle 9° 38'$	6·16	3·468	0·209	0·00902	0·00496	6·11	105 megohms.
b	5000	$286\angle 11° 4'$	3·112	1·531	—	0·0203	0·00555	3·055	—
c	5000	$579\angle 2° 50'$	3·15	3·57	4·04	0·0108	0·0039	—	—

In this case the attenuation of the loop circuit was near the calculated value in Table II., Chap. VIII., and this remark also applies to L and C and also Z_0. The value of R also agreed with the value measured by direct current.

The attenuation constant of the phantom circuit was 12 per cent. higher than that of the side circuit; the capacity was more than double that of the side circuit and the inductance less than half, which accounts for the increase in β.

In the case of the earth circuit, the relatively high inductance and large leakance are surprising, but four sets of tests were taken on four different wires, all of which gave very similar results. The earth circuit attenuation constant is 21 per cent. less than that of the loop circuit. There were a large number of other wires on the poles.

The circuits dealt with were four twisted wires spaced 12 ins. apart horizontally and vertically on the poles, and making a complete revolution every four poles.

Measurement of 300 lbs. Aerial Wires run straight and crossed every Half Mile.—In connection with the preceding tests a length of 63 miles of 300 lbs. double wire circuit run straight was tested in the same way as the twisted wires. The results were very similar to those obtained with the twisted wires. The difference in β at $\omega = 5000$, being 2 per cent. in favour of the straight wires. The overhearing arising from the twisted and revolved groups was similar, but there were only four wires on the poles in the case of the wires which were run straight.

217. Alternative Methods of Calculation and Measurement.—(*a*) *To find the real part of the Attenuation Constant.*—The preceding examples in connection with the Franke machine take into account the angle of the complex attenuation constant, and in cases where the circuit constants are required, this procedure is of course necessary. If, however, the real part of the attenuation constant only

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is required, the calculations may be somewhat simplified as follows:—

From formula (15) $e^{2\theta} = \frac{1+h}{1-h}$, where $h = \sqrt{\frac{Z_1}{Z}}$ and is a vector. If the vector h be expressed in the form $A \angle \phi$ then we may write

$$e^{2\theta} = \frac{1 + A \angle \phi}{1 - A \angle \phi}$$

Dividing the vector into its horizontal and vertical components we have

$$e^{2\theta} = \frac{1 + A\cos\phi + jA\sin\phi}{1 - (A\cos\phi + jA\sin\phi)}$$

For the attenuation constant β only the modulus is required, and we may therefore write

$$e^{2\beta l} = \sqrt{\frac{(1 + A\cos\phi)^2 + A^2\sin^2\phi}{(1 - A\cos\phi)^2 + A^2\sin^2\phi}}$$

$$= \sqrt{\frac{1 + 2A\cos\phi + A^2\cos^2\phi + A^2\sin^2\phi}{1 - 2A\cos\phi + A^2\cos^2\phi + A^2\sin^2\phi}}$$

$$= \sqrt{\frac{1 + 2A\cos\phi + A^2(\cos^2\phi + \sin^2\phi)}{1 - 2A\cos\phi + A^2(\cos^2\phi + \sin^2\phi)}}$$

and since $\sin^2\phi + \cos^2\phi = 1$, this becomes

$$= \sqrt{\frac{1 + 2A\cos\phi + A^2}{1 - 2A\cos\phi + A^2}}$$

$$= \sqrt{\frac{\frac{1}{A} + 2\cos\phi + A}{\frac{1}{A} - 2\cos\phi + A}}$$

$$\therefore e^{4\beta l} = \frac{\frac{1}{A} + 2\cos\phi + A}{\frac{1}{A} - 2\cos\phi + A}$$

$$\beta = \frac{1}{4l} \times \log_e \left(\frac{\frac{1}{A} + 2\cos\phi + A}{\frac{1}{A} - 2\cos\phi + A} \right)$$

(*b*) *Measuring the Electrical Constants of very short Transmission Lines.*—If a transmission line is electrically very short, say 0·1 mile of cable on a cable drum, the sending end impedance may be simplified as follows: With the distant end closed the sending end impedance is in the general case $Z_1 = Z_0 \tanh \theta$. If, however, the angle θ is very small the value of θ is practically the same as that of $\tanh \theta$. This may be verified by inspection of a suitable table of hyperbolic

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functions. In such a case $Z_1 = Z_0 \tanh \theta$ by (6), Chap. III., may be written $Z_1 = Z_0\theta$. But by (1) and (2), Chap. VI., since $\gamma l = \theta$

$$Z_0\theta = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \times \sqrt{(R+j\omega L)(G+j\omega C)} \times l = l(R+j\omega L) \quad (16)$$

If, therefore, the line is sufficiently short (say under $\beta l = 0{\cdot}1$), then the impedance measured with the distant end closed is the impedance of the conductor only.

If the distant end be open we have by (5), Chap. III.

$$Z = Z_0 \coth \theta = \frac{Z_0}{\tanh \theta}$$

and if the line is very short, then by the foregoing reasoning $\frac{Z_0}{\tanh \theta}$ is practically the same as $\frac{Z_0}{\theta}$. But

$$\frac{Z_0}{\theta} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \times \frac{1}{\sqrt{(R \times j\omega L)(G+j\omega C)}} \times \frac{1}{l} = \frac{1}{(G+j\omega C)l} \quad (17)$$

From (16) and (17) the constants R, L, G, and C may be calculated and the attenuation constant deduced thus. Dividing (16) by (17) we obtain

$$\theta^2 = (R+j\omega L)(G+j\omega C)l^2$$

The square root of this quantity is l multiplied by the complex attenuation constant.

The received current at the end of any uniform line without apparatus is by (10), Chap. III., $\frac{E_1}{Z_0 \sinh \theta}$, and for short lines this becomes $\frac{E_1}{Z_0\theta}$, since θ does not then differ appreciably from $\sinh \theta$; therefore from (16)

$$I_2 = \frac{E_1}{(R+j\omega L)l} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(*c*) *Fall of Potential Method for Measuring the Attenuation Constant in Long Lines.*—If the circuit is electrically very long there is no difference between the impedance measured from the sending end of the cable, whether the distant end is open or closed. In such a case the attenuation constant may be measured as follows. We know that in a long line

$$\frac{I_2}{I_1} = e^{-\beta l}$$

If, therefore, I_2 and I_1, the currents at the end and beginning of the line respectively, are measured by the Franke machine, and l is known, β may be deduced. This method assumes that a return loop is available. The sent current must, of course, be kept constant during the measurement.

218. Tests on an Unloaded Double-Wire Submarine Cable.—These tests show in a marked way the effect of capacity in producing both attenuation and distortion. The tests were taken by the Research Section, Engineer-in-Chief's Department, G.P.O., on a

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length of 4·826 nauts of double wire cable immersed in water in a tank. Experience shows that such measurements give practically the same results as cable submerged in the sea for double wire circuits.

Direct Current Measurements Wire to Wire.

Resistance, 22·8 ohms per naut.
Capacity, 0·182 mf. per naut.
Insulation Resistance, 1100 megohms per naut. Cores 1 and 3.
" " 540 megohms per naut. Cores 2 and 4.

TABLE IV.

ALTERNATING CURRENT MEASUREMENTS.

Mean of measurements on cores 1–3 and 2–4.

$2\pi f$	Characteristic impedance.		Complex attenuation constant.		Line constants per naut.				$\frac{G}{C}$
	Vector.	Angle.	β	α	R ohms.	L m.h.	G m.mhos.	C m.f.	
3000	221	−36° 1′	0·0654	0·0843	22·5	2·23	15	0·162	93
4000	194	−33° 27′	0·0717	0·1026	22·6	2·22	20	0·161	124
5000	179	−31° 7′	0·0774	0·1193	22·7	2·21	25	0·160	156
6000	165	−28° 59′	0·0817	0·1362	22·67	2·21	32	0·160	200
7000	156	−27° 0′	0·0852	0·1526	22·67	2·19	41	0·160	257

The only one of the constants R, L, G, and C which varied appreciably with frequency was G, and this varied considerably; a graph showing the variation is given in Fig. 125. For purposes of calculation it is usually accepted that the leakance varies directly as the frequency, but this curve shows some deviation from the straight line law (see para. 149, Chap. VIII.).

The resistance as measured by alternating current is very near the value obtained by a direct current measurement; the capacity, however, is less in the A.C. case. It is not, however, the author's general experience that any such marked difference is observed between direct current capacity measurements and tests at $\omega = 5000$. The single wire tests in Table V. illustrate this point, the difference is much less in that case. The attenuation constant of 0·0774 per naut or 0·0671 per mile for a conductor having a resistance of 19·6 ohms per mile loop is very high owing to the high capacity. In underground air-space cables a cable pair weighing only 70 lbs. per mile per wire, and having a resistance of 25 ohms per mile loop, has a much less attenuation, viz. 0·058. The variation of the attenuation constant with frequency in Table IV. is large, the difference between the β for $\omega = 5000$ and $\omega = 3000$ being 15 per cent. as compared with a difference of 4·5 per cent. in the case of the aerial lines in Table II.

219. Tests on a Single-Wire Submarine Cable.—The question as to whether a submarine telephone circuit on a single wire is practicable as a general rule, is one that has attracted much attention in

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the past. (See an article by Monsieur Devaux-Charbonnel in "La Lumière Electrique," 8 Feb., 1913, p. 165, "Resistance et affaiblissement des lignes à Simple fil," also an article by the writer in the *Post Office Electrical Engineers' Journal*, July, 1914, entitled "Two Loaded Phantom Circuits in a Four-Wire Submarine Cable.")

As a matter of fact it has since been demonstrated that there is

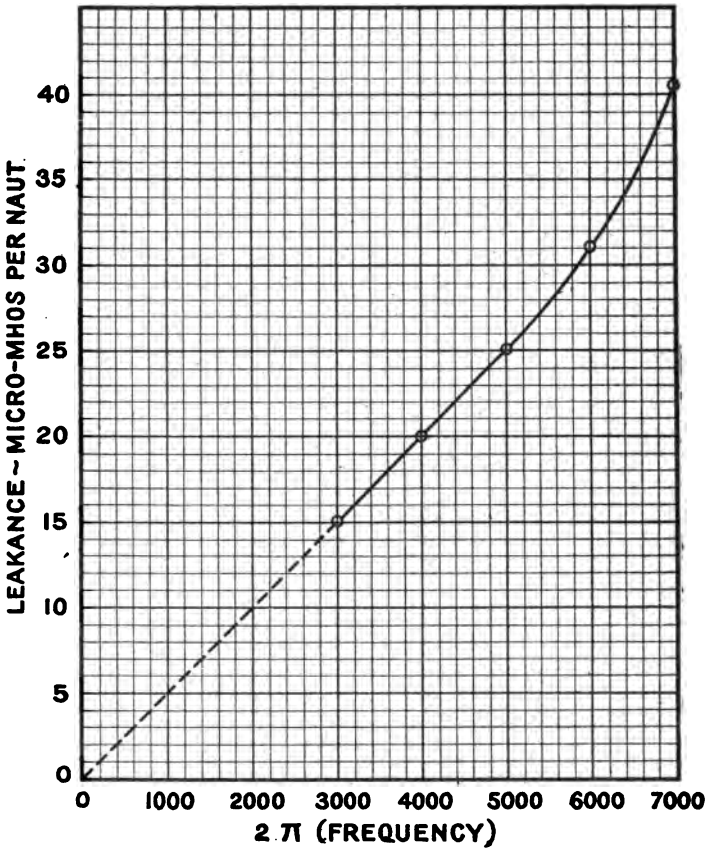

FIG. 125.—Variation of Leakance with Frequency in Double-Wire Submarine Cable.

no difficulty in working such a circuit (see para. 202, Chap. IX.). It may be remarked that any inductive disturbance noticed on a single wire is usually limited to the cable hut, and disappears if a separate earth wire is run out to sea for say half a naut. In order to obtain information as to the effects of alternating currents on a single-wire telegraph cable protected by an iron sheath a series of tests was carried out by the staff of the Engineer-in-Chief to the Post Office in 1912, and the results of the alternating current tests are given in

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Table V. The most striking feature of the tests is the increase of effective resistance with frequency. At $\omega=5000$ this is 13·36 ohms against 11 ohms as measured by direct current. Similar results

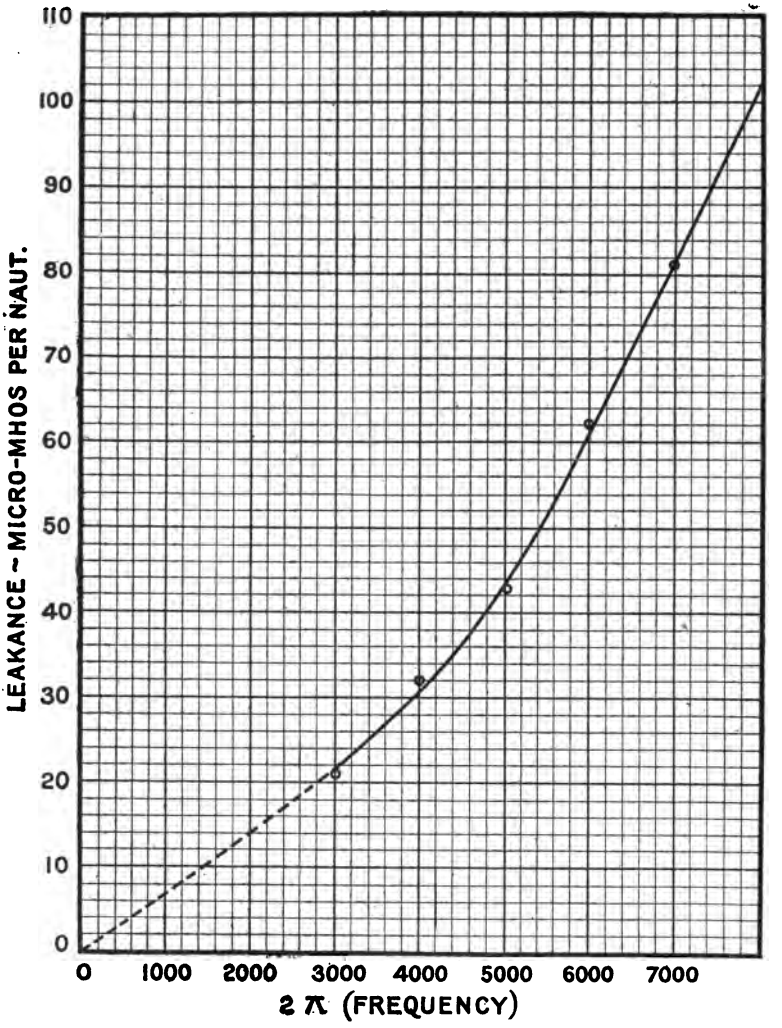

FIG. 126.—Variation of Leakance with Frequency in Single-Wire Submarine Cable.

have been obtained by M. Devaux-Charbonnel in the tests referred to. The leakance also varied considerably with frequency as shown in Fig. 126 and did not follow a straight line. The A.C. capacity is slightly less than the D.C. capacity. The increase in resistance is

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probably due to the effect of the external field of the single wire acting on the iron sheath and sea water and giving rise to energy losses (see para. 142).

Before laying a single-wire telephone cable it would be advisable to make tests on a sample length of the proposed cable to determine its effective constants.

TABLE V.

TESTS ON SINGLE-WIRE SUBMARINE CABLE.

Alternating Current Tests. Summary of Results at Different Frequencies.

$2\pi f$	Characteristic impedance.		Complex attenuation constant.		Complex attenuation constant.		Line constants per naut.				$\frac{G}{C}$
	Vector.	Angle.	Vector.	Angle.	Real pt.	Imag. pt.	R ohms.	L m.h.	G m. mhos.	C m.f.	
2000	138	−36° 24′	1·46	+53° 52′	0·0525	0·0715	12·0	1·84	—	0·31	—
3000	120	−33° 17′	1·86	+55° 27′	0·0645	0·0935	12·70	1·72	21	0·315	67
4000	108	−31° 2′	2·23	+57° 10′	0·074	0·1130	13·08	1·62	40	0·315	126
5000	99	−28° 53′	2·25	+59° 35′	0·079	0·1340	13·36	1·58	43	0·316	137
6000	94	−27° 16′	2·84	+60° 48′	0·085	0·1510	13·60	1·50	62	0·306	200
7000	90	−25° 16′	3·11	+61° 44′	0·090	0·1670	13·90	1·44	81	0·316	256

Direct Current Tests.

Resistance per naut	11·0 ohms
Capacity per naut	0·327 m.f.
Insulation resistance per naut	2570 megohms
Electrification steady.	

Both the single-wire and double-wire submarine cables in respect of which tests are given in Tables IV. and V. had the same weight of copper conductor and guttapercha per naut, and the following comparison of the measured constants is therefore of interest:—

Table of Comparative Results from Tables IV. and V. at $\omega = 5000$.

	Double-wire circuit.	Single-wire circuit.
Characteristic impedance .	$179 \angle 31° 7'$ ohms	$99 \angle 28° 53'$ ohms
Attenuation constant (β) .	0·0774	0·079
Resistance	22·7 ohms	13·36 ohms
Inductance	2·21 mh.	1·58 mh.
Leakance	25 micromhos	43 micromhos
Capacity	0·16 mf.	0·316 mf.
$\frac{G}{C}$	156	137

The preceding details of tests on transmission lines refer to unloaded circuits. Details of tests on loaded circuits are more conveniently given in Chap. IX.

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The Thermionic Valve used as an Oscillator.—This can be used as a portable sine-wave generator. For description see end of Chap. XIV.

220. Tests on Artificial Cables.—Artificial cables for telephone work such as are used for standard cable tests are generally made up of resistance and capacity, the maximum unit being that shown in Fig. 127. Thus, a section of 10 miles of the standard cable (see Chap. XI.) is made up of 5 two-mile sections, each section having the same resistance and capacity as the real line. The reason for this may be understood from a consideration of the equivalent π or T circuit (App. III. and IV., Figs. D and G in each case). It will be seen that the mathematical quantities $\tanh \theta$, $\sinh \theta$ and $\coth \theta$ which govern the construction of the equivalent circuit, are such that with small values of θ there is practically no difference in the numerical value of θ and of the corresponding functions $\sinh \theta$, $\tanh \theta$, and $\coth \theta$,

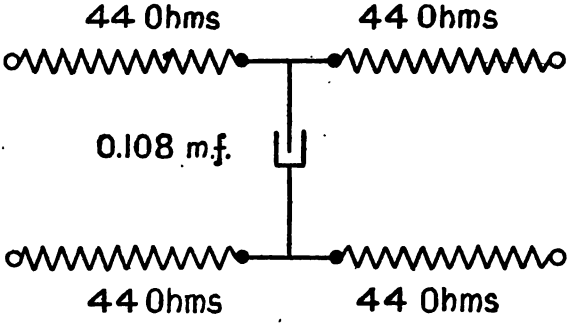

FIG. 127.—Two-Mile Unit of Artificial Standard Cable.

and it follows that an artificial circuit made up of sections which represent small line angles, will approach nearer and nearer for all possible purposes to the real line as regards its impedance, and attenuation at all frequencies. An equivalent circuit equal to 2 miles of unloaded standard cable is equal to a smooth line for practical purposes, and this accounts for the usual method of construction. (See Chap. IV., para. 77.)

For any one frequency, an equivalent circuit may be calculated and made up on the principles shown in Chaps. IV. and VI., and it will be equivalent for any line angle, for all electrical operations conducted from its two ends, to a real circuit having the same line angle. It does not by any means follow, however, that the equivalent circuit will have the same electrical properties as the real line for other frequencies than the one for which it is calculated; in fact, experiment and calculation show that this is far from the case. It is to be expected therefore that the waves generated by the voice in such a line will suffer some distortion; but the extent and practical importance of this point is a matter for experiment and comparative voice tests. Tests on an equivalent circuit of one section representing 20 miles of standard cable and another representing 40 miles of standard cable have been made in the Post Office laboratory as follows:—

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The circuit shown in Fig. 72, which is theoretically equivalent to 20 miles of standard cable (for the constants of this cable see Table II., Chap. VIII.) at a frequency such that $2\pi f = 5000$ was set up, and measurements made with the Franke machine. The following results were obtained :—

$\omega = 2\pi f$	Characteristic impedance, Z_0	Attenuation constant, β
4000	$525 - j\,477$	0·0845
5000	$414 - j\,381$	0·1080
6000	$366 - j\,312$	0·1250
7000	$346 - j\,268$	0·1340

These results are shown plotted in Fig. 128, and the calculated figures for standard cable are added in the form of dotted curves.

It will be seen by Fig. 128 that the attenuation constant and also the real and imaginary parts of Z_0 at an angular velocity of $2\pi f = 5000$ do not in any case vary more than 2½ per cent. from the standard cable curves. There is, however, a much greater variation in the impedance of the two cables over a range of frequency varying from $\omega = 4000$ to $\omega = 7000$ radians per second. The variation of the attenuation lengths of the two cables with frequency over the same ranges also show a considerable variation, the artificial cable varying much more with frequency than the real one. In order to test the practical equivalence of the two circuits, the standard cable and the equivalent circuit were compared by a speaking test. It was found that the volume

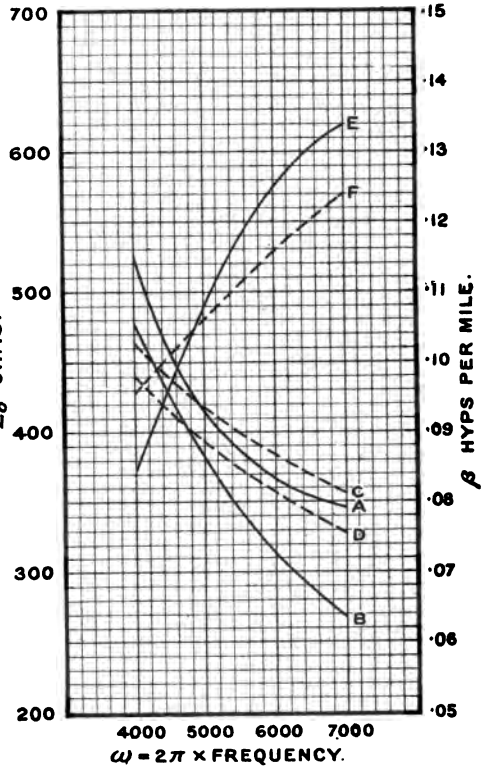

FIG. 128.—Comparison of Equivalent 20-lbs. Circuit with Standard Cable. Variation of Z_0 and β with Frequency.

A—Z_0 (Real) of Equivalent Circuit.
B—Z_0 (Imaginary) of Equivalent Circuit.
C—Z_0 (Real) of Standard Cable.
D—Z_0 (Imaginary) of Standard Cable.
E—β of Equivalent Circuit.
F—β of Standard Cable.

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of speech was equal to 20 miles of standard cable for which it was designed. This result was the mean of a number of tests made by three observers. The articulation and pitch of speech were similar in the two cases; in fact, the difference in this respect could only be detected by one of the three observers who took part in the test. When, however, the equivalent circuit which had been designed to be equal to 40 miles of standard cable was tested, there was found to be a very marked difference in the pitch of speech when speaking over the two cables, the equivalent circuit having the higher pitch, and this made accurate comparison difficult. The matter was therefore not proceeded with except to verify that the desired attenuation constant at $\omega = 5000$ had been correctly arrived at by calculation and experiment.

It may be further stated as a general conclusion from these and other experiments that if the attenuation and impedance of an artificial line are required to be equal to those of a real uniform line over a wide range of frequencies, it is a necessary condition that the equivalent artificial circuit should be made up of T or π sections having a relatively small line angle, otherwise the requirement cannot be met in all respects. In other words, the more exacting the requirement, the more must the artificial circuit approach the constitution of the real line, *i.e.* it must approximate to uniform distribution throughout its length.

221. Artificial Circuits to meet Special Requirements.—It is sometimes required to construct *impedances* which shall be equal over a wide range of frequencies to the impedance of a real transmission line, or combination of line and apparatus. It may be that such an impedance is only required for balancing purposes in a telephone relay set and that the question of the attenuation constant is of no importance. In such cases equivalent circuits of the T or π type may be too complex for the required purpose, and simpler methods are adopted. A description of a method of calculation in such a case is given in Chap. XIV.

Again, for some local purpose an artificial cable may be required where a fixed attenuation length is required. The following details relate to a case where an approximate value of 20 miles of standard cable was required for making speaking tests and a relatively large number of the cables were to be provided. It was considered that for this particular requirement a simpler arrangement than the equivalent T or π circuit would suffice, and the following steps were taken to obtain a suitable arrangement at a cheap cost. It will be seen by Fig. 72 that the true equivalent circuit involves the use of inductances which it was desired to avoid. In this case a nominal circuit such as that dealt with in Chap. III., Fig. 10 and Fig. 38, were utilised to make up the cable, the values being chosen as follows:—

(*a*) For 20 M.S.C. the value of $\beta l = 2{\cdot}123$, hence

$$\frac{\text{received current}}{\text{sent current}} = e^{-2{\cdot}123} = 0{\cdot}1197$$

Now try a nominal circuit of 19·1 M.S.C. for which $\beta l = 2{\cdot}025$ (see Fig. 129), in order to see whether this meets the requirement. We have

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$$\frac{\text{received current in nominal circuit}}{\text{received current in equivalent circuit}} = 0{\cdot}907 \text{ (see Fig. 38)}$$

but $$\frac{\text{received current in equivalent circuit}}{\text{sent current in equivalent circuit}} = \epsilon^{-2{\cdot}025} = 0{\cdot}1316$$

$$\therefore \frac{\text{received current in nominal circuit}}{\text{sent current in equivalent circuit}} = 0{\cdot}907 \times 0{\cdot}1316 = 0{\cdot}1194$$

Hence the received current at the end of a nominal circuit of 19·1 M.S.C. should be practically the same as that with 20 miles of actual standard cable, if the impedances of the equivalent and nominal circuits are equal.

Now 19·1 mls. × 88 ohms per ml. = 1681 ohms
and 19·1 mls. × 0·054 mfds. per ml. = 1·032 mfds.
so that the nominal circuit is as shown in Fig. 129.

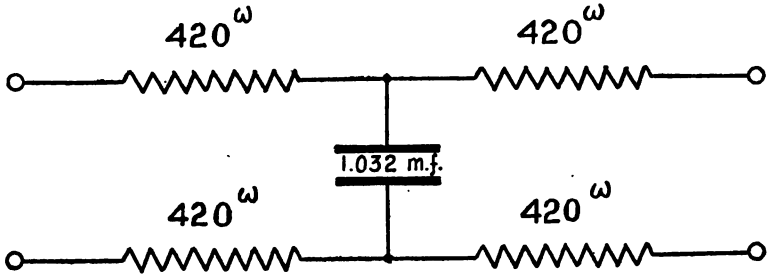

FIG. 129.

This circuit (Fig. 129) was set up and a speaking test gave its transmission equivalent as 20 M.S.C.; the articulation, however, was clearer than that of the cable. Its impedance was measured by a Franke machine with a current of 1 m.a. and $\omega = 5000$; the open impedance was found to be $864 \angle 13° 0'$ and the closed impedance $880 \angle 11° 55'$.

(*b*) It will be seen from the foregoing that the impedance of the circuit in Fig. 129 is higher than that of actual standard cable, and this fact limits its use as a standard in speaking tests. An attempt was made to make its impedance as nearly equal to standard cable as possible by adding capacity at each end, and the circuit shown in Fig. 130 was set up. By inserting different values of x it was found that when $x = 0{\cdot}2$ mfds. that the open and closed impedances, when measured as before, were $577 \angle 46° 40'$ and $589 \angle 47° 30'$ respectively. These values are approximately the same as those for standard cable. A speaking test gave the transmission equivalent of Fig. 130 when $x = 0{\cdot}2$ mf. to be 22 M.S.C. and the articulation was identical with that of the cable. It was not important in this case whether the cable had an exact value of 20 M.S.C. or one or two miles in addition so long as the correct value was known, and the values which had been found experimentally were therefore adopted. It was found possible to utilize cheap stock items to make up the

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values of resistance and capacity, viz. 400 ohms, 1 mf. and 0·2 mf. capacity. This case is dealt with at length because it shows a cheap and practicable way of making up a length of artificial standard cable, if the requirement is not too exacting, and it illustrates one use of Fig. 38, Chap. IV. It may be added that a number of artificial cables were made up in this way, and they were found to be practically alike.

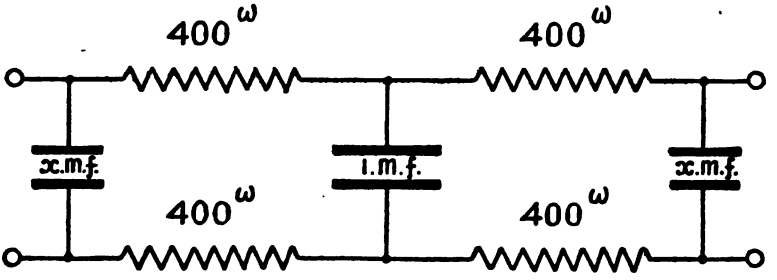

FIG. 130.

222. The Cross-Talk Meter.—This apparatus is made to the design of the Western Electric Company, and is manufactured so far as this country is concerned by Mr. R. W. Paul, London. As its name implies the apparatus is used for cross-talk tests, the method of its

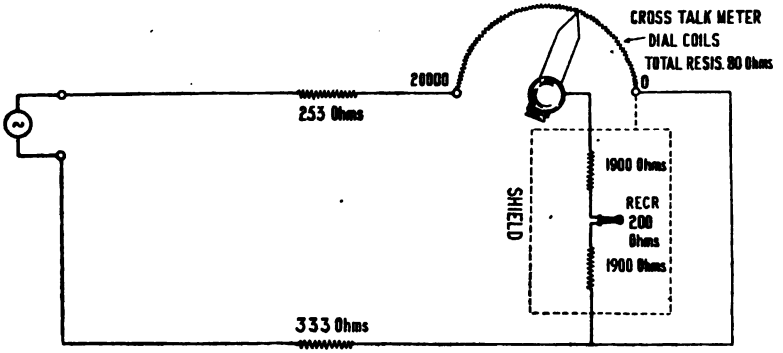

FIG. 131.—Cross-Talk Meter. Explanatory Diagram.

use will be better understood after the details of its construction have been explained. A simplified diagram of the instrument is shown in Fig. 131 and the arrangement of the coils and indicating dial is given in Fig. 132. The function of the meter is to shunt any required fraction of the speaking current from the receiver as may be seen by Fig. 131. If the current leaving the transmitter is represented by 1,000,000 then the figures as engraved round the dial, as shown in Fig. 132, give the number of millionths of the outgoing current passing through the receiver when the rotating arm is opposite the required figure. It should be carefully noticed that

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the instrument does not measure the magnitude of the current, but only indicates the fraction of it passing through the receiver.

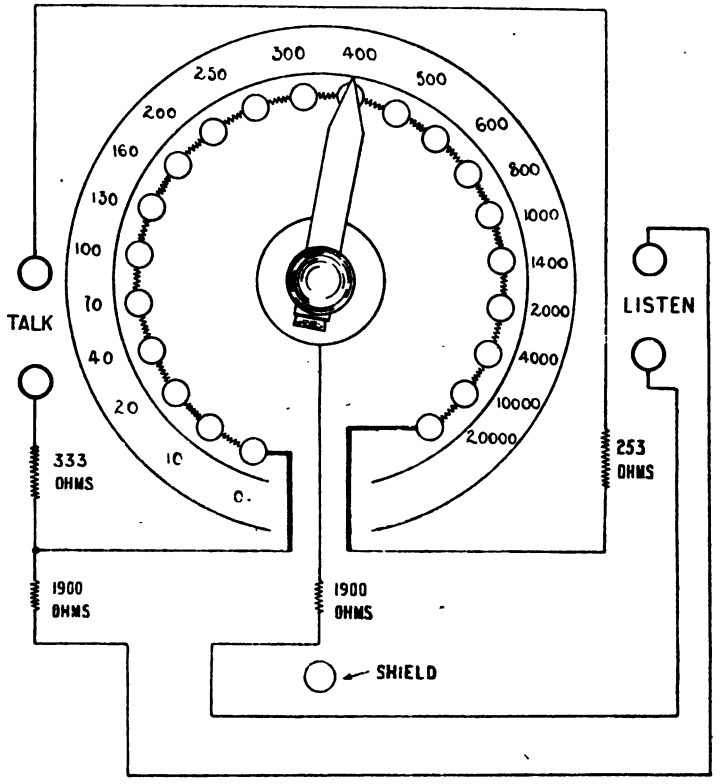

FIG. 132.—Cross-Talk Meter. Arrangement of Dial and Coils.

The values of the shunt resistances in relation to the figures on the dial are as follows :—

Dial numbers. Col. 1.		Col. 2.	Resistance.	Equivalent standard cable value for figures in col. 1.
Between 20,000	and	10,000	40·0 ohms	36·9
,, 10,000	,,	4,000	24·0 ,,	43·45
,, 4,000	,,	2,000	8·0 ,,	52·1
,, 2,000	,,	1,400	2·40 ,,	58·6
,, 1,400	,,	1,000	1·60 ,,	62·0
,, 1,000	,,	800	0·80 ,,	65·2
,, 800	,,	600	0·80 ,,	67·2
,, 600	,,	500	0·40 ,,	70·4
,, 500	,,	400	0·40 ,,	71·7
,, 400	,,	300	0·40 ,,	73·7
,, 300	,,	250	0·20 ,,	76·5

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Dial numbers. Col. 1.		Col. 2.	Resistance.	Equivalent standard cable value for figures in col. 1.
Between 250	and	200	0·20 ohms.	78·3
,, 200	,,	160	0·16 ,,	80·3
,, 160	,,	130	0·12 ,,	82·4
,, 130	,,	100	0·12 ,,	84·5
,, 100	,,	70	0·12 ,,	86·9
,, 70	,,	40	0·12 ,,	90·3
,, 40	,,	20	0·08 ,,	95·5
,, 20	,,	10	0·04 ,,	101·9
,, 10	,,	0	0·04 ,,	108·6

Referring now to Fig. 131, suppose the rotating arm is in the position 0, there is then $253 + 80 = 333$ ohms in circuit as shown in the upper part of the diagram, and 333 ohms as shown in the lower part of the diagram; further, the receiver is short-circuited. If the arm is in the position marked 20,000 and the receiver has an impedance of 200 ohms to the A.C. voltage in the circuit, we then have $2 \times 1900 + 200 = 4000$ ohms in the receiver circuit, and 80 ohms shunting it by means of the dial coils; the current passing through the receiver is therefore $\frac{80}{4000}$ of the total current $= \frac{x}{1000000}$, where $x =$ the scale reading on the dial, in this case $x = 20{,}000$. When the arm is on the 10,000 stop the 4000 ohms in the receiver is shunted by 40 ohms, and we then have $\frac{40}{4000} = \frac{x}{1000000} = \frac{1}{100}$, and so on. It is evident that the shunt is only strictly correct if a suitable receiver and frequency are used. It may be remarked that the standard 60ω receiver has an impedance of approximately 200 to 210 ohms, at a frequency of 800 cycles per second, which is the usual frequency for telephonic measurements.

When the instrument is at zero the total resistance in circuit at the terminals of the transmitting device is 666 ohms, which is the impedance of a heavy gauge aerial line. When the arm is moved to 20,000 the total resistance is 664·4 ohms, so that the total resistance is practically constant.

The reduction of volume which takes place in the receiver with various values of shunts may be likened to the reduction of volume which occurs by attenuation when a current is transmitted along the standard cable, and it is of considerable assistance in interpreting the results to know the relative amount of standard cable which would give the same reduction in volume as any given number of millionths of the outgoing current; this information is given in the preceding table.

The full shunt is in operation when the rotating arm is on the 20,000 stop. The total resistance of all the coils between 20,000 and 0 is 80 ohms, and as the arm moves from 20,000 round towards zero, it cuts out of the shunt all the resistance between 20,000 and the position of the moving arm. If this eliminated resistance is called x, then the number of millionths indicated on the circular scale at the point opposite the moving arm is in all cases

$$\frac{1000(80 - x)}{4} = \begin{cases} \text{millionths of total current passing} \\ \text{through the receiver.} \end{cases}$$

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Method of using the Instrument. — The method of using the cross-talk meter is the same in principle as that utilised in the case of the standard cable. A switch connects a receiver with a circuit of some kind which includes a source of cross-talk, and the magnitude of the disturbance is noted by the ear. The same receiver is then switched into the cross-talk meter, and the intensity of sound reproduced by varying the shunt across the receiver. The source of disturbance, whether it be human speech through a telephone, or some other source, such, for example, as a "howler," is generally used to produce the disturbance in the circuit under test and in the cross-talk meter. The howler, however, will generally give higher results than speech tests, and in important cases the latter should be used as a check test.

In view of the fact that the meter only registers millionths of the current passing through the receiver, and not its absolute magnitude, it follows that the circuit giving rise to the cross-talk should have the same impedance as the cross-talk meter, if the recorded result on the cross-talk meter is to be strictly correct. Notwithstanding this, however, the instrument is very useful in practice even in cases where the

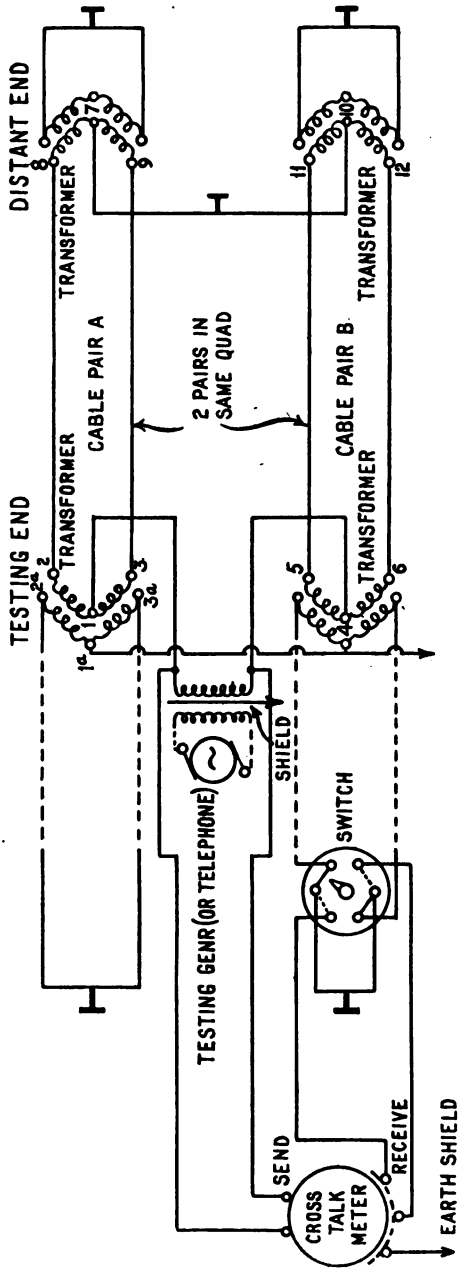

FIG. 133.—Cross-Talk Test on Phantom Circuit in Cable. Meter and Circuit in Parallel.

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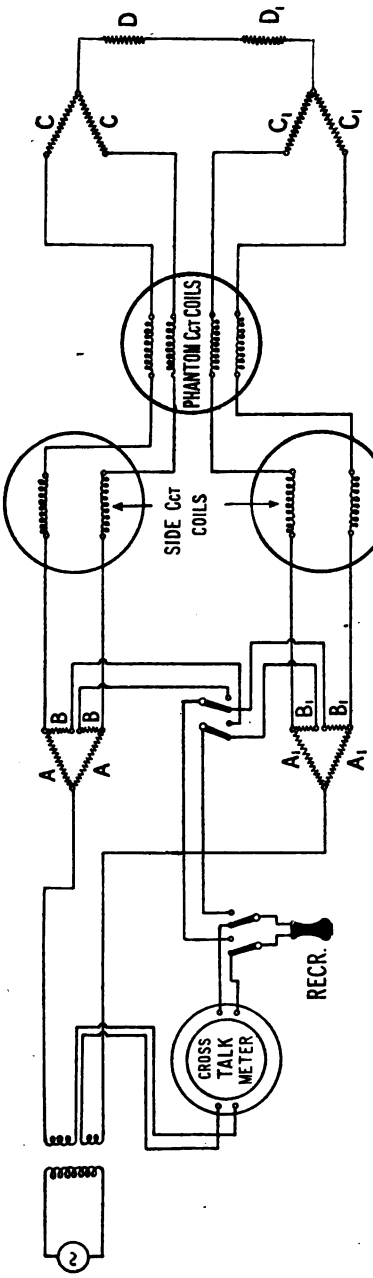

FIG. 134.—Cross-Talk Test for Loading Coils.

A, A_1, B, B_1, C, C_1, and D, D_1 are Resistances of suitable Values to represent Ordinary Circuit Conditions.

impedances of the circuit under test and the cross-talk meter are not the same. This arises from the fact that engineers accustomed to such tests know by experience that a given result on the cross-talk meter is associated with certain definite results in working conditions, which are permissible if the figures do not exceed a certain amount which varies with the circumstances and is known. See, for example, the details of tests on loading coils in Chap. IX.

In testing underground cable circuits for cross-talk, a common arrangement is to join the circuit under test for cross-talk, and the cross-talk meter, permanently in parallel during the test, as shown in Fig. 133. This ensures that the same voltage is applied to the meter, as to the circuit under test, irrespective of their impedances. If the circuit length should be altered in the test, as may be desirable, this arrangement keeps the voltage at the sending end the same both for the meter and the cable. If it is desired to test with equal currents, as may, for example, be sometimes desired in testing loading coils where the inductive effects largely depend on current, the object may be achieved by increasing artificially the impedance of the circuit of lowest impedance until the testing circuits are of equal impedance.

As an alternative the

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cross-talk meter is sometimes placed in series with the circuits under test, in this case both the meter and the circuit receive the same current, but cases may arise where this arrangement is not sufficiently sensitive. If the impedance of the cross-talk meter, for example, is much greater than that of the cable, the voltage at the cable end will be relatively small, and a relatively small volume of sound will result.

If it is desired to ascertain the amount of cross-talk in commercial conditions in any given circuit, the circuit under test is usually closed through an impedance such as is used in practical working—1000 ohms is a convenient value; it represents the impedance of a common battery set in circuit with a long subscribers' cable circuit. If the distant end were closed without a resistance, the voltage at that end would be nil, and the result misleading on a short line.

In testing very long circuits a normal condition would be obtained by having the distant end open. The current entering any circuit under test may be made the same as that in a long line by closing the distant end through an impedance equal to the characteristic impedance of the circuit, and this is frequently done.

In order to detect cross-talk due to capacity unbalance, the most severe test is that made with the distant end open. This is because such unbalance increases with the voltage, and the voltage in the line is at its maximum with the distant end open. If the unbalance is due to current effects, say the effects of current on loading coils giving rise to cross-talk, the most sensitive condition is that with the distant end closed.

Fig. 134 shows the usual arrangement for testing loading coils for cross-talk.

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CHAPTER XI

THE STANDARD CABLE AND ITS USES, WITH APPLICATIONS OF TRANSMISSION CALCULATIONS TO PRACTICAL CASES

223. The Standard Cable and its Uses.—It has already been demonstrated in Chap. V. that the sounds of the human voice are produced by highly complex vibrations, and illustrations of this complexity are there given. As a consequence, a number of cases occur in practice where technical information of great value in telephonic transmission can only be obtained by comparative speech tests. There are, in fact, properties of the voice which cannot, up to the present, be imitated by a testing machine having any known combination of definite frequencies, as will be evident when the following properties of speech are recalled. Of course considerable use is made of machine tests as shown in Chap. IX., but the need for supplementary tests will now be dealt with. All voices have four important characteristics which are of special importance in telephony. These are—

(*a*) Volume or intensity.

(*b*) Articulation or clearness of speech. Perfect telephonic articulation assumes not only that such speech is sufficiently clear to be easily understood, but that all the frequencies of the voice are preserved in their relative proportions as impressed upon the transmitter.

(*c*) Pitch.

(*d*) Timbre or quality.

Now the *articulation and quality* of the voice cannot be adequately represented and judged except by speech tests.

Volume can often be predicted from machine tests, but there are numerous cases where such is not the case, as will now be seen.

The *pitch* of a voice is often materially modified by passing through telephonic lines and apparatus, and this effect is best judged by the ear.

Effects of Frequency on Apparatus.—The effect of frequency on many kinds of apparatus is very marked, and there are numerous cases where tests at one frequency would give no idea of the result obtained by speech. For example, the diaphragms of both transmitters and receivers tend to resonate at different frequencies, as explained in Chap. V., and the properties of such apparatus require speech tests to fully demonstrate them. The electrical efficiency of transformers and condensers is much greater to relatively high frequencies than to low ones. The impedance of signalling apparatus

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bridged across telephone lines varies largely with frequency. The matter is considerably complicated if a number of these items are included in the same telephone circuit, as is often the case, and there is no single frequency or combination of frequencies which can adequately replace the human voice as a means of correctly finding the combined volume, articulation, and pitch of speech in such cases. The volume can, however, in many cases be predicted by machine tests and calculations, at a single frequency, and this is of great importance. Attempts have been made to more accurately represent the effects of the voice than has hitherto been possible by means of machine tests, by using several simultaneous frequencies. See a paper on "Telephonometry," by B. S. Cohen, in the professional papers of the Institution of Post Office Electrical Engineers.

Effects of Frequency in Telephone Lines.—As already explained the frequency of 800 cycles per second has been adopted as a mean frequency for speech, and this usually permits of the possibility of predicting the volume of speech when considered independently of apparatus effects, in any length of line. Indeed, for purposes of design of circuits, particularly in cables, it is absolutely essential, if correct results are to be obtained, that information of the ***measured*** effective value of the circuit constants at some definite frequencies should be available to give data for estimates and specifications. It is still, however, highly desirable to make speech tests to determine the character of the articulation, etc. The attenuation in lines can often be determined by the standard cable method, which also affords a ready means of determining the terminal loss when a suitable circuit length is available. Speech tests are made by means of a standard cable, and its uses may be defined as follows.

224. Principal Uses of a Standard Cable.—These are: (*a*) To determine the standard cable equivalent of telephone apparatus and combined apparatus and lines (such, for example, as common battery subscribers' circuits of various lengths); (*b*) to determine the standard cable equivalent of exchange wiring and apparatus generally (such, for example, as cord circuits); (*c*) to find the standard cable equivalent of terminal loss and transition loss in circuits in cables and open lines, and to find also the approximate standard cable equivalent of such circuits.

Limitations of the Standard Cable Method.

(*a*) The constants of circuits such, for example, as R, L, G, and C, cannot be found by this means.

(*b*) If the "tone" of the circuit to be compared is very different to that of the standard, accurate comparison of the relative volume of sound in the two cases is very difficult, but it is usually possible for practised observers to arrive at an approximately correct result in such cases.

(*c*) The personal factor of the observers making the tests may to some extent affect the result. This is, however, guarded against in England by taking the mean values of tests by three observers. Experience shows that some observers consistently obtain a little higher or a little lower result than others.

It may be finally mentioned that the work tends to become

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monotonous, particularly if the observers are not keenly interested in the results. On the other hand, the actual observations need not necessarily be taken by skilled engineers; intelligent youths, who need not have a high degree of technical training, suffice for the purpose, if the work is closely supervised and verified by a competent and experienced engineer.

225. Definition of Standard of Transmission.—The measurements outlined in the preceding paragraphs involve a transmission standard. Just as weights and distances are reduced to pounds and miles so it is advantageous in telephone transmission to have a standard.

Case (*a*): To express the attenuation of speech currents due to their transmission through lines, or a combination of line and apparatus, in terms of the length of a standard cable which produces the same attenuation as the lines and apparatus in question, when the latter, and the standard cable, are in turn joined to the standard transmitting and receiving apparatus, and the same volume of speech is impressed upon the standard transmitter in both cases. Equality of attenuation exists when the same volume is received in the standard receiver, from the two combinations which are in turn connected to it. The equality is judged by the ear of the listening observer at the receiving end of the circuit. (Fig. 142 and the test it illustrates furnish a simple example of this definition.)

Case (*b*): To express the transmission efficiency of any combination of lines and apparatus, including transmitting and receiving apparatus (when this is not the same as the standard apparatus) in terms of the length of standard cable connected to the standard instruments, which produces the same volume of speech in the standard receiver as is produced in the receiver of the combination under test when the same volume of speech is alternately impressed on the standard transmitter, and the transmitter of the combination under test. (Fig. 140 and the test it refers to illustrate this.)

226. Description of Transmission Standard.—This is a telephone circuit consisting of two common battery subscribers' telephone sets, including 300 ohms resistance, connected by a cable as shown in Fig. 135. A similar standard was first used in America.

The Standard Telephone Set consists of (*a*) a 60-ohms bell receiver; (*b*) a solid-back transmitter; (*c*) an induction coil with windings having resistances of 17 ohms and 26 ohms respectively; (*d*) a 2 mf. condenser; (*e*) a magneto bell of 1000 ohms resistance.

The subscribers' line is represented by an ohmic *resistance* (*g*) of 300 ohms.

Exchange apparatus included in the speaking circuit.—The telephone set is connected as shown in Fig. 135 to (*f*), a supervisory relay of 70 ohms resistance shunted by a non-inductive resistance coil of 30 ohms; (*h*) a battery of 24 volts (secondary cells), which is not allowed to fall below 22 volts; (*k*) a toroidal coil with four windings, each having a resistance of 22·5 ohms.

The Standard Cable is a cable containing conductors weighing 20 lbs. per mile per wire, and having the constants shown in Table II., Chap. VIII. This cable is somewhat similar to a number of cables which are largely used in local circuits in commercial practice, and it is of convenient size for obtaining a considerable length of cable circuit loop in a relatively short length of cable, by doubling the

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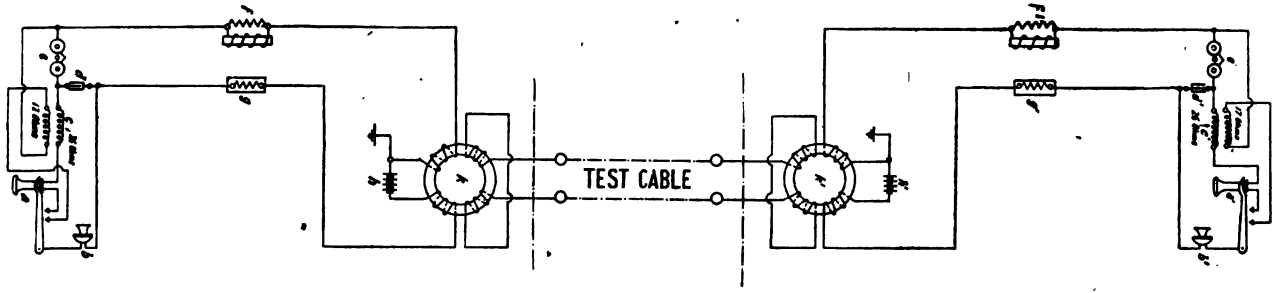

FIG. 135.—Transmission Standard adopted by the Post Office and formerly also by the National Telephone Co.

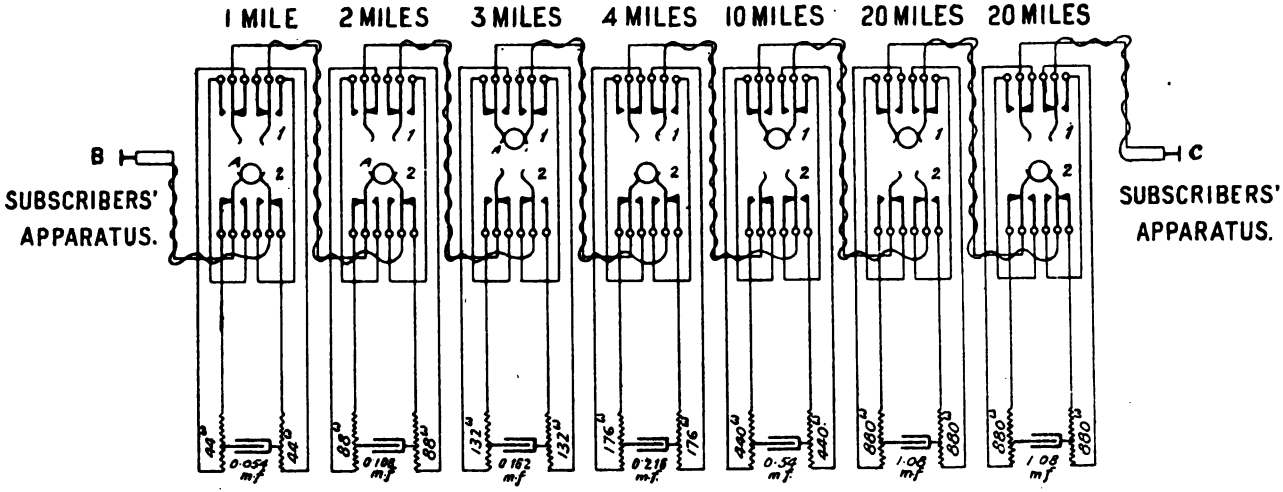

FIG. 136.—Diagram showing Switching Connections of Standard and Artificial Cables.

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circuit back, These facts probably account for its adoption as a standard. The standard cable is so arranged that its length can be rapidly varied to give any length—say between one mile and 60 miles—by steps of one mile. A theoretical switching arrangement for such a cable is shown in Fig. 136. The position of the cable keys may be changed from position 1 to position 2 or *vice versâ*, but they cannot have an intermediate position. According to the position of the key the line controlled by it is included in the speaking circuit or withdrawn from it. In position 1 the cable is out of circuit, in position 2 it is in circuit. As shown in the illustration, 27 miles of standard cable is in circuit for speaking between B and C. When any of the keys are in position 2 the lengths of cable controlled by them are in series.

Selection and use of Transmission Standard Apparatus.—In the first instance the apparatus of which the standard speaking set is made up, particularly the transmitter and receiver, are selected so as to have good but not exceptional qualities of volume, articulation, and tone. The standard set when so selected (particularly the transmitter and receiver) should be carefully preserved, and only used for exceptional reference purposes. Heavy currents must be carefully prevented from passing through the transmitter and receiver. It is advisable that six secondary transmitter and receiver standards, at least, as near to the standard as possible, should be carefully selected and labelled, with their relative values, and preserved for use. These standards should be carefully compared with each other at regular intervals, and if any one of them becomes defective it must be replaced. By this method the permanence of the speaking standard is as nearly as possible preserved, and experience shows the method to be fairly effective. Experiments have, however, been made with electromagnetic transmitters and non-polarised receivers to secure standards which vary as little as possible and the results are promising.

227. Zero Loop Standard and Sending and Receiving Allowance Curves.—When the resistances g and g_1 in Fig. 135 are zero, we have the equivalent of a common battery subscribers' line of no resistance; this is usually called "zero loop," and in such a case the speaking range of the apparatus is at a maximum. If a length of cable is inserted in the sending subscribers' loop the speaking range of the apparatus is reduced, but the same volume of speech as before may be obtained in the receiving apparatus, if the length of standard or test cable is suitably reduced. For example, if the standard apparatus is used on zero loop at each end, and speech be transmitted through 47 miles of standard cable, a certain volume of sound is heard in the receiver at the receiving end of the line. If now a subscribers' circuit made up of $3\frac{1}{2}$ miles of standard cable be inserted in place of g, Fig. 135, in the sending instrument, and the local receiving circuit be unaltered, it is necessary to take out of circuit a little over 10 miles of the standard test cable, and thereby reduce it to about 37 miles of standard cable in order that the volume of speech may be the same as that observed when zero loop was the condition at the sending end. Similarly, an equivalent transmission value or "sending allowance" may be found for any length of subscribers' circuit, as compared with transmission on zero loop.

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We proceed thus: (*a*) Lengths of standard cable having all resistances from 0 to 500ω are tested in the local line as described; (*b*) The resistances 0 to 500 ohms are then plotted on squared paper, against the number of miles of standard test cable by which the total length of standard test cable had to be reduced in the case of any given resistance of local line, in order to obtain the same received volume of speech with the same impressed volume at the sending end. A useful curve is thus obtained which gives the "*sending allowance for local lines.*"

This curve for the 24-volt system, which is shown with curves for other voltages in Fig. 137, shows the effect so far as "sending" is concerned of the resistance in the lines of common battery subscribers. For some less important tests ohmic resistance is used

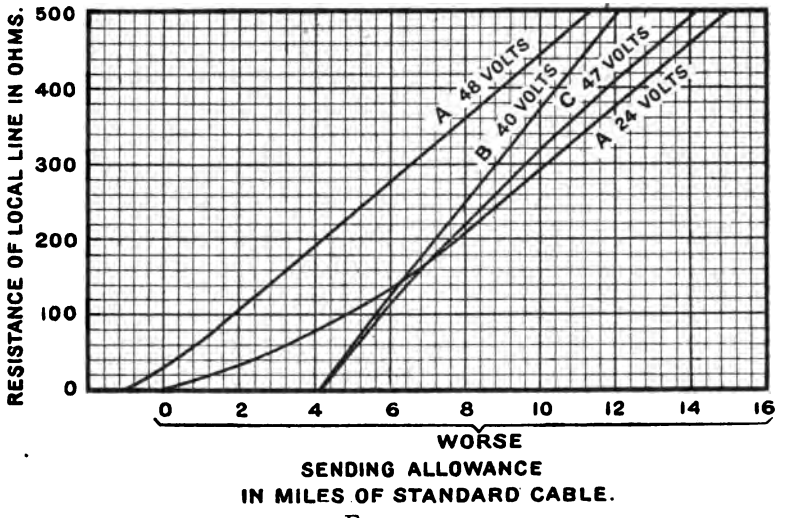

FIG. 137.

instead of cable; as shown at *g*, Fig. 135. All subscribers' lines have, however, of course, capacity as well as resistance, but so far as "sending" is concerned, the capacity has a very small effect in subscribers' lines in common battery loops made up of small gauge cable, if the resistance does not exceed 300 ohms. This is probably due to the relatively large effect of the inductance in the exchange and subscribers' apparatus. It must not, however, be assumed that the subscribers' line acts merely as a resistance for other purposes, as its impedance is very different to that of mere resistance. The curve shows the result of tests on a subscriber's circuit made up of 20 lbs. standard cable.

Receiving Allowance Curve.—If the subscribers' circuit at the sending end be kept constant, and "zero loop" be the arrangement at the receiving end, we have the condition for maximum volume of speech at that end, in so far as that depends on changes in the resistance of the subscribers' loop at the receiving end. If, however,

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a length of subscribers' cable (say 3½ miles) be substituted in the receiving loop for the zero loop, the volume of speech will be reduced, but the original maximum volume due to zero loop can be restored by reducing the length of standard or test cable between the standard instruments (see Fig. 135) until the required equality of volume is obtained. The amount by which it is found necessary to reduce the length of the test cable is the " Local Line Receiving Allowance " of the 3½ miles in the receiving subscribers' circuit, and similarly the

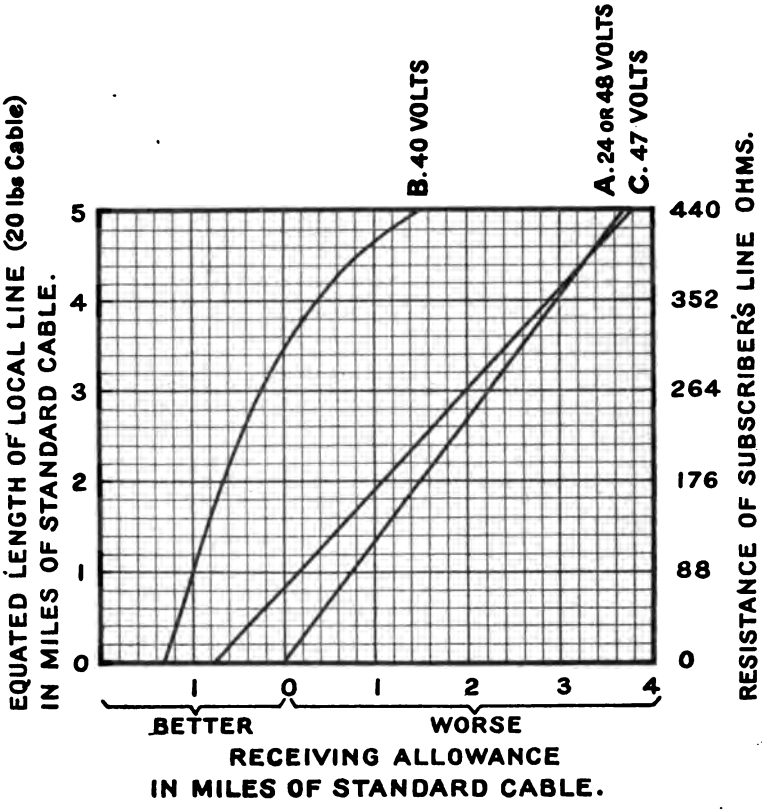

FIG. 138.

" Receiving Allowance " of any and all lengths of the receiving cable may be found. If all lengths of cable between zero loop and 5 miles of this cable are compared in this way a curve similar to that marked 24 volts as shown in Fig. 138 is obtained. For example, if the volume of speech with zero loop at the receiving end is noted when speaking over a length of 47 miles of standard cable (test cable), and two miles of cable be then inserted in the receiving circuit, the volume of speech will naturally be reduced, but if at the same time 1½ miles is taken out of the test cable it will be found that equality of volume is

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produced; this means that 47 M.S.C. + zero loop is equivalent to $45\frac{1}{2}$ M.S.C. + 2 M.S.C. loop, and therefore $47 - 45\frac{1}{2}$ or $1\frac{1}{2}$ M.S.C. is the receiving allowance for 2 miles of standard cable in the receiving subscribers' circuit. These figures are given in Fig. 138 where 2 miles of standard cable in the vertical column or ordinate is connected by the curve with $1\frac{1}{2}$ miles in the horizontal line or abscissa, and the receiving allowance for any other length may be similarly found. In all cases the "allowance" is less than the miles of cable in the test circuit; probably owing to the effect of the apparatus at both ends of the relatively short local line acting in conjunction with the capacity in the line.

228. Current Supply Loss Curves.—Every subscriber's line which is made up of a given kind of cable has a definite attenuation length which is numerically the β of the cable as given in Table II., Chap. VIII., multiplied by the length of the subscribers' line, and this is the theoretical receiving allowance for that length of line and type of cable, in the absence of the modifying effect of the apparatus at the two ends of the local line. Owing to this latter effect, however, the receiving allowance varies to some extent from the theoretical line allowance, as will be seen by comparing the experimental allowances in Fig. 138 with the values calculated as just explained. The loss obtained experimentally is often termed the "high frequency loss" (this term is also used to denote the standard cable equivalent of bridged apparatus, etc., on lines), and this loss forms a portion of the "sending allowance" as well as constituting the receiving allowance. In fact, the sending allowance loss is made up of the high frequency loss, plus a loss due to the continual reduction of current supply to the transmitter, as the line resistance increases with the length of the subscribers' line. This loss is known as the "current supply loss," and in any given common battery system it depends only on the resistance of the line. For example, take the case of the standard apparatus joined to some definite length of local subscribers' circuit made up of standard 20 lbs. cable, or of 10 lbs. cable. The current supply loss is found by subtracting the receiving allowance, or high frequency loss, for that length, from the sending allowance for the same length in each case, and the difference depends on the line resistance only. If a length of 10 lbs. cable is selected which has the same resistance as the standard cable (the lengths of the two cables in this case of course being different), the difference between the receiving and sending allowances will be the same in the two cases, as it would indeed be in the case of any other conductor having the same resistance, when the standard circuit apparatus is used. It follows that if the receiving allowance curve is known for any type of circuit the sending allowance curve can be found by adding the receiving allowance curve to the current supply loss curve, or conversely the current supply loss curve may be found by subtracting the receiving allowance curve from the sending allowance curve. Fig. 139 has been prepared in this way by subtracting the receiving allowance curves in Fig. 138 for 24, 40, 47, and 48 volts respectively, from the sending allowances curves in Fig. 137 for the same voltage values.

For example, the sending allowance in the 24-volt system for a subscriber's circuit having a resistance of 300 ohms is 10·2 miles

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of standard cable. The corresponding local line in the case of the receiving allowance curve, Fig. 138, is a subscriber's line 3·41 miles in length, for the standard cable has a resistance of 88 ohms per mile and $88 \times 3{\cdot}41 = 300$ ohms. The receiving allowance for this line by Fig. 138 is 2·5 M.S.C., and the current supply loss is therefore $10{\cdot}2 - 2{\cdot}5 = 7{\cdot}7$ M.S.C. This will be found to be the figure given in Fig. 139.

If the precise receiving allowance loss is not known, an approximate sending allowance may be found by taking the attenuation length of the local line, no matter what its gauge may be, and adding the value obtained from Fig. 139 for the current supply loss associated with the known resistance of the local line and the voltage of the common battery system in question. In this way one curve for one given voltage system with resistances as ordinates and loss in standard miles as abscissæ will suffice for all types of

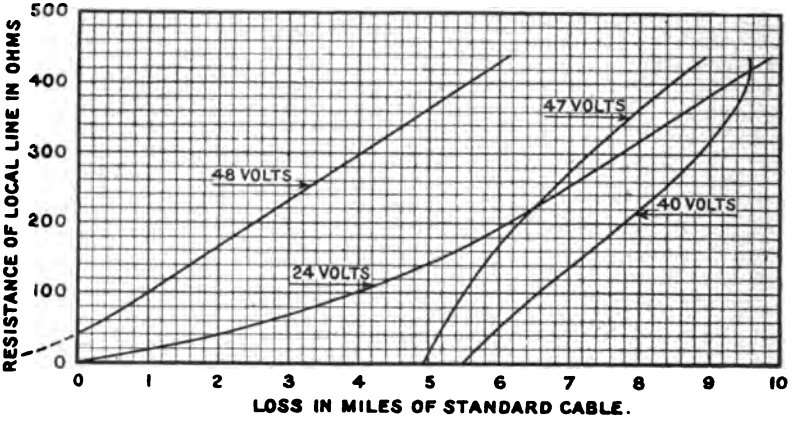

FIG. 139.—Current Supply Loss.

subscribers' cables. Figs. 137 and 138 also approximately represent, however, a mean for all types of local lines whether overhead or underground, and are used in a general way by the Post Office as the allowances for all such lines. Similarly, Fig. 139 gives the approximate current supply loss for local lines of all types based on their resistance. In admitting such approximations as sufficiently accurate it must be borne in mind that different common battery transmitters would in any case give local line sending allowance curves which would vary more than the sending and receiving allowances of different lines vary when their transmission efficiency is assumed to change with line resistance only. In arriving at the sending allowance curve the mean of the results of tests on a number of common battery transmitters was taken into account. The resistance of all exchange cords, heat coils, etc., should always be added to the line resistance in computing sending allowances, and the high-frequency loss resulting from the introduction of these items should be taken into account.

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It should be borne in mind that curves such as those given in Figs. 137 and 138 depend largely on the type of cord and subscribers' apparatus used in the test as well as on the voltage, and if any of these values change the curve may change. The curves are given to illustrate the method of making them up, and they at the same time give a general idea of the effect of voltage, but the apparatus used may be changed from time to time.

229. Effect of Fixed Resistance in Subscribers' Standard Transmission Circuit, and Range of Telephonic Speech with Standard Apparatus.—As shown in Fig. 135, a resistance of 300 ohms is included in the speaking apparatus at both the sending and receiving ends of the test cable. As previously stated this forms part of the standard transmission circuit. At the sending end of the circuit the resistance has the effect of reducing the range of speech transmission by 10·2 M.S.C. as compared with the transmission efficiency of the sending apparatus on zero loop. In the case of zero loop the allowance is 0.

At the receiving end of the line the 300 ohms ohmic resistance in the transmission standard set has the effect of reducing the volume of received speech by 1·5 M.S.C. as compared with the received volume when zero loop is used. The receiving allowance with zero loop is 0. In practical work it is advantageous to express the standard of transmission in terms of zero loop.

It is found that commercial speech can be conducted without difficulty by means of the standard apparatus with zero loop at both ends of the test cable, when the latter is constituted by 47 miles of the standard cable. A quiet room at both ends and speakers with suitable voices are stipulated as necessary conditions. It follows that if the standard apparatus has 300 ohms inserted at each end of the line the same volume of speech at the receiving end of the line with the same impressed volume at the sending end necessitates the reduction of the length of standard cable in the test cable from 47 to 35·3 miles, for we now have by the previous definitions—

Sending allowance	10·2	M.S.C.	that is βl	$= 1{\cdot}0812$
Receiving ,,	1·5	,,	,,	$= 0{\cdot}1590$
Test cable ,,	35·3	,,	,,	$= 3{\cdot}7418$
	47·0	,,	,,	$4{\cdot}9820$
			Say βl	$= 5$

230. Strength of Direct Current in Standard Local Circuit.—If the current circulating in the circuit is known and also the type of the circuit, the sending allowance can be found. On the other hand, it is possible to say what current is required in order that a given allowance may be obtained. The following shows the method, although the individual items may change.

The resistance due to the exchange apparatus is as follows in accordance with the details already given:—

Toroidal repeating coils	45 ohms.
Resistance of supervisory relay	21 ohms.
Extra resistance in office wiring	7 ohms.

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Resistance of subscriber's apparatus in sending circuit, *i.e.* solid back transmitter and induction coil winding	r_1 ohms.
Resistance of subscriber's loop	R ohms.
Minimum voltage permissible	22 volts.

The direct current circulating in the circuit is therefore

$$I = \frac{22}{66 + R + r + r_1}$$

The current measured from the external cable terminals at the exchange is, if r is neglected,

$$\frac{22}{66} = 0{\cdot}33 \text{ amp.}$$

If the subscriber's instrument is added on zero loop we have to add the resistance of the transmitter—say 60 ohms—+the line winding of the induction coil (17) = 60 + 17 ohms. The formula then becomes

$$I = \frac{22}{66 + 77} = \frac{22}{143} = 0{\cdot}154 \text{ nearly}$$

If a 300-ohms loop is added we have the standard circuit, then $I = \frac{22}{443} = 0{\cdot}050$ amp. or 50 milliampères nearly.

If the resistance of the apparatus at the subscriber's station is excluded we have $443 - 77 = 366$.

In that case $I = \frac{22}{366} = 0{\cdot}06$ amp. or 60 milliampères.

If, therefore, we measure the current at the terminals of the line in this subscriber's station and know the exchange voltage, the resistance is directly obtained, and knowing that the exchange resistance is 66 ohms, it is evident that a 300-ohms line circuit is measured. The sending allowance can then be obtained from Fig. 137. A similar equation for each voltage may be made.

231. Applications of the Standard Cable Method of Testing.—Transmission losses due to apparatus in bridge and in series in transmission lines. The following illustrates the method of using the standard cable for this purpose. Suppose, for example, it were desired to find the effect on the received volume of sound of bridging two similar indicators of relatively low impedance across the middle of a length of the unloaded test cable in Fig. 135. This would have the effect of altering the received current as compared with the current received when the bridges were absent. The effect of the bridges in such a case would be determined by finding how much of the standard test cable had to be switched out of circuit with the indicators bridged across it, in order to produce the same received volume of sound in the receiver as was observed at that point when the bridges were absent. The amount of standard cable withdrawn would be the "allowance" for the bridges. On the same principle the allowance for condensers in series, transformers, cord circuits, etc., may be found. *Strictly speaking all such transmission losses apply only to the circuit in which they are observed,*

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as the result depends on the relative impedance of the apparatus and line. The results obtained, however, may be taken as an average figure applying to other circuits which do not differ largely from the unloaded standard cable in impedance. In cases where the conditions differ materially from those in which a given loss is found this must be taken into account. It is usual in well-equipped transmission laboratories to have artificial cables to represent more than one type of line. The Post Office, for example, has an artificial line to represent aerial lines (see article on "An Artificial Equivalent of an Open Wire Line for Telephonic Experiments," by G. M. B. Shepherd in the *Post Office Electrical Engineers' Journal*, Oct., 1914), and means are also provided for loading the standard cable. Not only so but if the conditions are exceptional a special artificial cable may be made up on the principles developed in Chaps. IV., VI., X., and XIV., and this is in fact often done in the Post Office Laboratory. Any details of transmission losses which are determined by test and tabulated for general use should specify the general conditions in which they may be used. Thus losses due to apparatus bridged across the middle of a heavily loaded circuit will be different to the same apparatus bridged across the same unloaded circuit.

Method of determining the Transmission Efficiency of different Common Battery Transmission Systems in Terms of the Standard C.B. System.—The standard transmission circuit may be used to test other well-known common battery systems of transmission. When an occasion arises which necessitates the comparison of the transmission efficiency in such cases the testing arrangement is that shown in Figs. 140 and 141. Suppose, for example, it were required to compare a second common battery system for transmission efficiency with the standard transmission system for long-distance purposes, say over a length of 47 miles of the standard cable on zero loop. It will be seen from Fig. 140, which represents this case, that two lengths of standard cable can be switched into either system at the will of the controlling testing officer. A preliminary test of the two systems would be advantageous in the first instance, using the cable of fixed length only, and switching it into each circuit alternately by the switches at *a* and *b*. The speaker at the sending end usually counts one to five in a uniform tone over each circuit alternately, whilst at the same time the receiving observer carefully notes the difference in the received volume of speech. In all probability speech over one of the systems would be louder than that over the other system in these circumstances. The variable balancing cable is now switched into the system which gives the louder volume of speech, and kept there for the time being, the cable of fixed length still being used for comparing both systems as before. A length of cable is next inserted in the balancing cable and varied until speech over the system in which it is included is of equal loudness to speech on the other system. The length of standard cable in the balancing cable indicates that the circuit in which it is placed gives louder transmission than the other one by the number of miles of standard cable in the balancing circuit. In a similar way to the preceding tests on zero loop, a comparison can be made for any length of subscribers' circuit. It would be simpler if a separate standard cable

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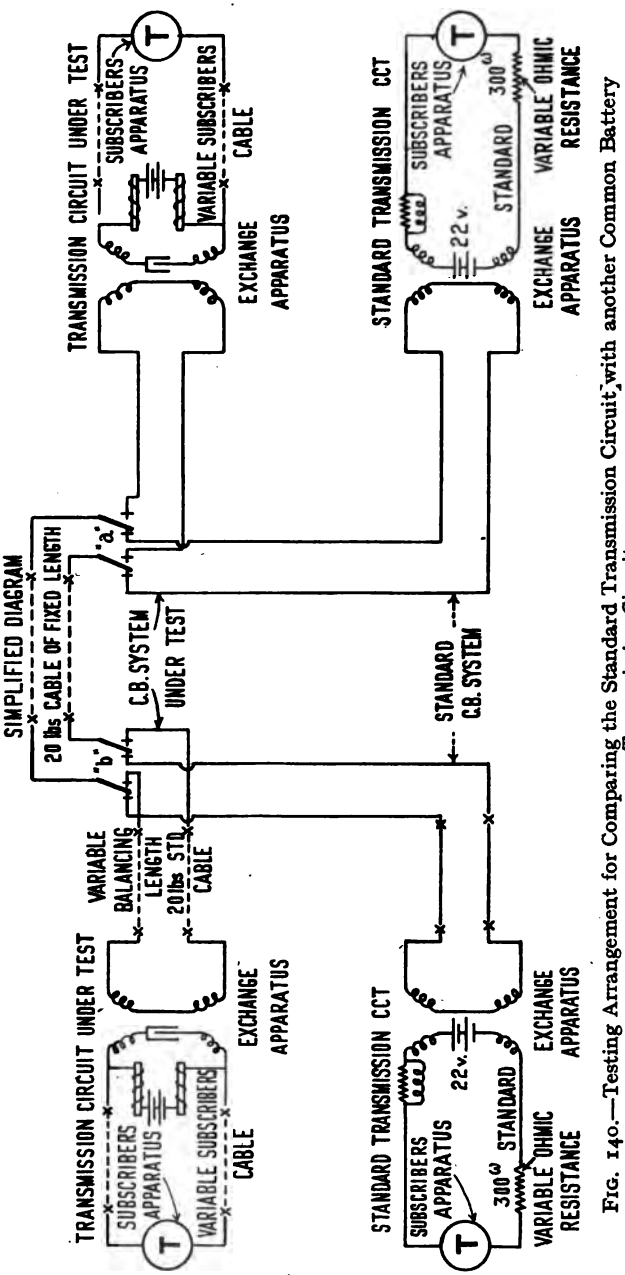

FIG. 140.—Testing Arrangement for Comparing the Standard Transmission Circuit with another Common Battery Transmission Circuit.

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were inserted in each circuit and the length of one of them altered until equality of volume in the receiving apparatus was noted; but there would be a risk of the testing cables not being quite equal; this risk is reduced to a minimum in the method adopted, as the balancing cable is usually of small length, and any error would be confined to that length. The total length of cable in use is also considerably reduced as compared with the use of a separate cable for each circuit. The switching movement of the balancing cable is shown in Fig. 141; it is omitted for simplicity in Fig. 140. The two

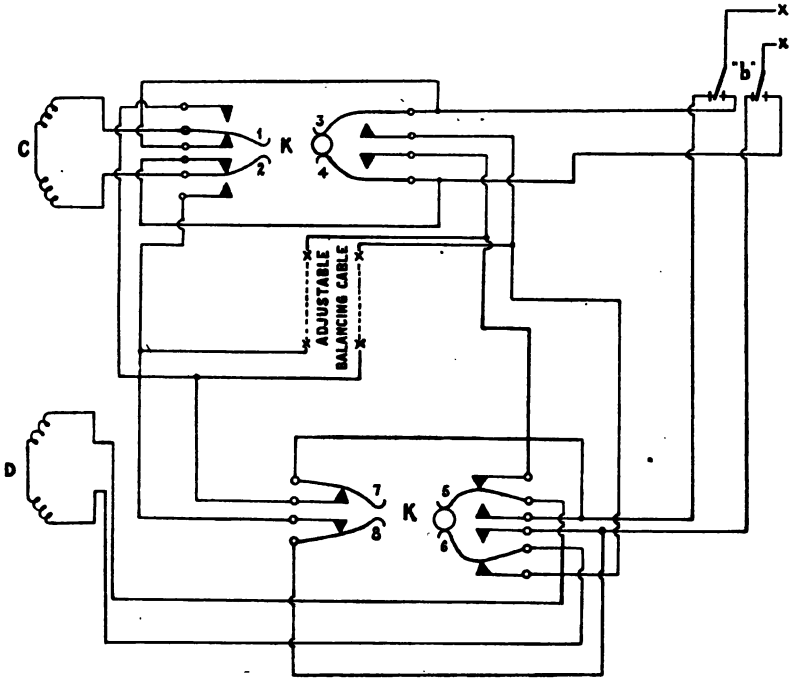

FIG. 141.—Switching Movement controlling the Balancing Cable shown in Fig. 140. The Balancing Cable is here shown in the Standard Transmission Circuit.

key levers shown at K, Fig. 141, are fastened together and move always in the same direction, and this permits of easy manipulation. The changes from one condition to another should usually be as rapid as possible, otherwise the accuracy of the observations may suffer. This remark applies to all standard cable tests and a rapid switching device should always be considered.

A test for articulation would, of course, be necessary in the comparison of two C.B. systems; this should be made with both short and long lengths of line, speech being conducted both softly and loudly.

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Having completed the test of the whole system, the transmitters, receivers, cords, etc., could be compared with those in the standard circuit, keeping everything constant except the item tested. It would, of course, be necessary to bear in mind that any given item might be designed for use on its own system only, and would not necessarily give the best results on the standard circuit. Notwithstanding this fact, experience shows that such comparisons often bring useful facts to light.

Figs. 137 and 138 show the result of a comparison of four common battery systems with various lengths of local line. The comparison was made in the Post Office Laboratory, and the curves are taken from an article by Cohen and Hill in the *Electrician* for July, 1916.

The Relative Transmission Efficiency of Telephone Conductors.—The following is given as a simple illustration which it is necessary to understand, in order to follow subsequent explanations.

If two uniform circuits not differing greatly in impedance and of sufficient length be compared by a speech test as shown in Fig. 142,

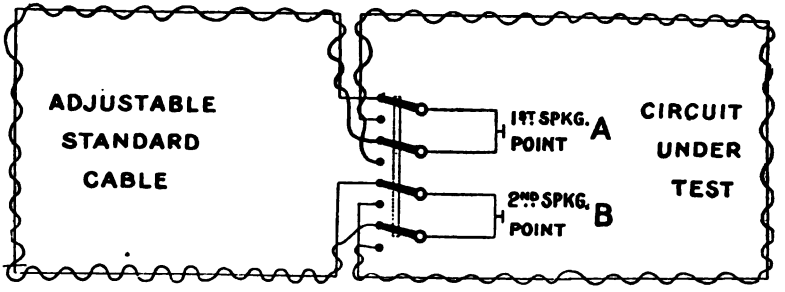

FIG 142.
Diagram showing Method of Comparing Circuits.

it is found that when the speaker at point No. 1 speaks in a level tone alternately on each circuit by means of the change-over switch, and the observer at speaking point No. 2 finds the volume of sound to be equal on the two circuits by a listening test, then the ratio between the lengths of the two circuits is always the same. Thus if 20 miles circuit length of one cable gives speech of equal loudness to 10 miles of the other, then equality of speech will be obtained with 30 miles of the one circuit and 15 of the other, also with 40 miles of the one and 20 of the other. This fact may be expressed as follows:—

$$\frac{\text{length A}}{\text{length B}} = \text{A constant quantity (for equal transmission)} \quad . \quad (1)$$

This result may be mathematically deduced from formula (16), Chap. II. In fact, if the so-called attenuation constants (β) per mile are calculated for any two circuits from their electrical constants at 800 periods per second, the same ratio is obtained as in speech tests, if the lines do not differ in impedance.

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Standard Cable Equivalent of Terminal Loss in Loaded Cables.—The results of tests such as those in the preceding paragraph may also be graphed, and a straight line graph is obtained as shown in Fig. 143 (lower curve). The figures graphed are those on page 276. In practice, all lines are compared with the standard cable. The attenuation value or standard cable equivalent of this line is taken as unity. The advantage of the graphical method is seen when the characteristic impedances of the lines tested are not equal to that of the standard cable. In that case we have terminal loss, as already explained in Chap. VII. The attenuation law still holds good for the current as it flows along the line. In such a case the loss of current due to end effects, which may be regarded as quite separate from the attenuation loss, would be a fixed amount, no

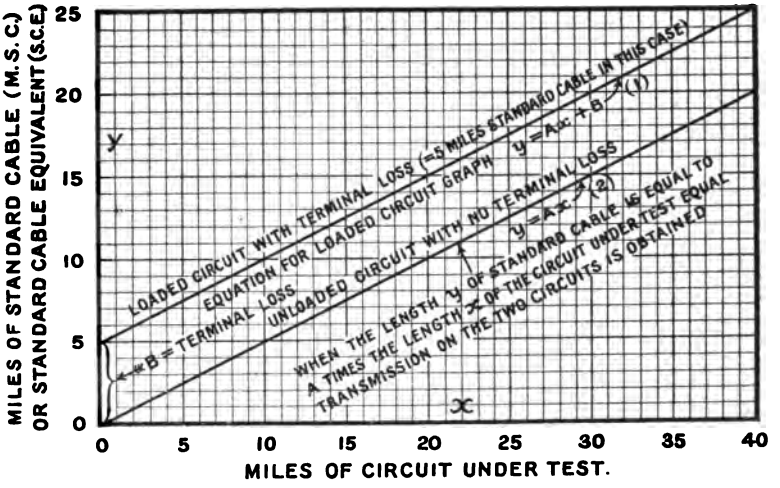

FIG. 143.—Comparison of Loaded and Unloaded Circuits by Speech Tests. Explanatory Graph.

matter what the length of the line (if it were long enough), and this loss could be expressed as equal in effect to the attenuation which would take place in speaking over an ascertained length of the standard cable. This might be called the standard cable equivalent of the terminal loss. It may be shown graphically, as in Fig. 143 (upper curve). See also formula (61), Chap. VII. The terminal loss in Fig. 143 is the S.C.E. between the point where the loaded graph cuts the vertical and zero; the loaded graph is parallel to the unloaded line graph in this case, because the attenuation constants of the loaded and unloaded lines are assumed to be the same for simplicity. This means that the loaded circuit is of less gauge than the unloaded one. If the gauges were the same the graphed lines would cross.

Terminal loss can be reduced to a very small quantity by various devices.

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(1) If speech over the long loaded lines is conducted from long local unloaded lines joined to them (say long unloaded common battery subscribers' lines 4 miles in length) the impedance conditions are considerably modified and the terminal loss is considerably reduced. (See Fig. 82.)

(2) If it is not eliminated a terminal transformer may be used to complete the desired reduction. These transformers are made so that the impedances of their windings are similar to the lines to which they are connected, and this reduces the loss to a minimum. With the most suitable transformers the loss can usually be brought below 0·5 S.M. if the terminal transformer replaces an ordinary

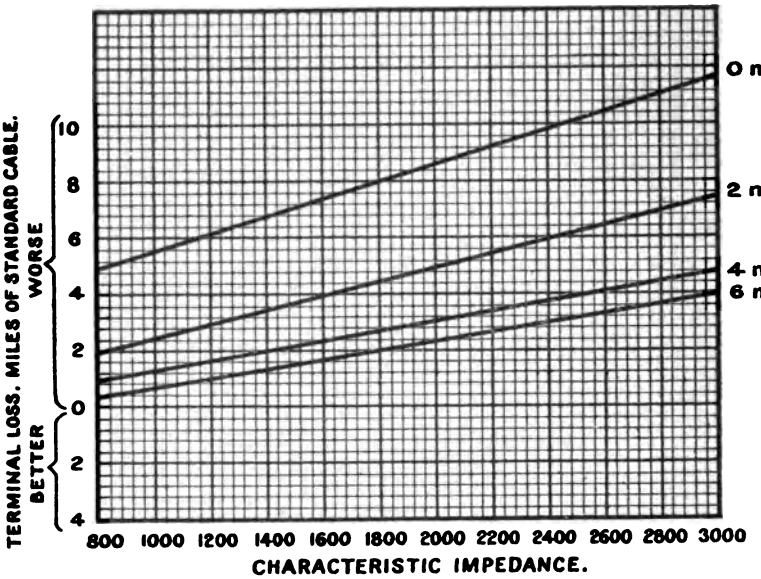

FIG. 144.—Variation of Terminal Loss with Line Impedances.

Standard Circuit with different lengths of Subscribers' 20-lbs. Cable, joined at both ends to a long loaded line.

repeater or transformer. If it does not the transformer core loss must be added.

These points are illustrated in Figs. 144 to 150, which give terminal loss curves for a number of useful cases. If the reader has any doubt as to the interpretation of these curves the remarks on Figs. 80 to 83 should be again consulted. If terminal loss curves for 10 lbs. cables on loaded side circuits are required, the method of interpolation described in connection with Fig. 83 is useful for obtaining an approximate curve. It should be remembered in this connection that the terminal loss value for zero loop is the same for a curve of terminal loss in connection with any local line, seeing that there is really no line in circuit. Zero loop values for a large

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range of impedances are given in Fig. 144, and these can be used to obtain the commencing point for interpolation on the curve for

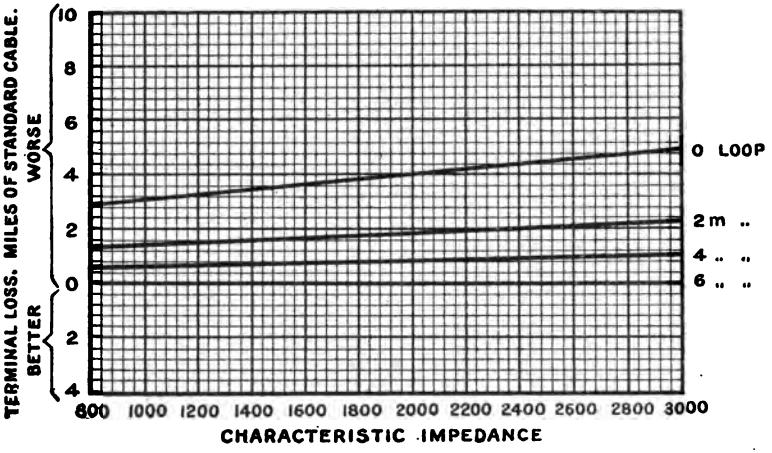

FIG. 145.

Same as Fig. 144, but with Terminal Transformers replacing the Repeating Coils in the Standard Circuit.

terminal loss on 10 lbs. cables in Fig. 146. Any difference which may be observed in zero loop curves in different graphs is due either to differences in the lines and apparatus tested or to experimental errors. Figs. 144 and 145 in this chapter were published in an article by Mr. B. S. Cohen in the *Post Office Electrical Engineers' Journal* for April, 1913.

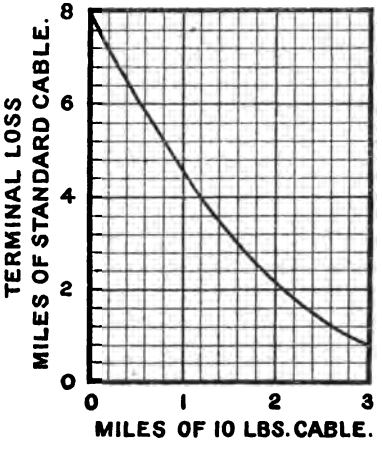

FIG. 146.—Terminal Loss at both ends of Heavily Loaded Side Circuits using Standard Apparatus and 10 lbs. Subscribers' Circuits. Normal spacing.

232. Transmission Schemes. —*General Remarks.*—The object of a transmission scheme is to provide engineers with information which enables them to design plant in conformity with a predetermined transmission standard in given areas, and between fixed points, at a minimum cost. The careful preparation of such a scheme is therefore of great importance.

The class of service to be provided determines to some extent the grade of the transmission standard, or in other words the loudness of the talk to be provided: thus in the case of stations in the

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same local area or in the case of towns not far distant from each other, where the conversation has to be carried on in some cases in noisy offices subject to disturbance from street traffic, or other causes, it would be reasonable and practicable to provide a louder standard of speech than in the case of the longest distance trunk service. It would indeed be reasonable to expect that a quiet receiving room or a silence cabinet should be provided to take advantage of the necessarily small volume of speech which electrical and physical limitations impose in the case of the longest trunk connections. If such facilities are not provided this class of service is impossible. The two cases supposed are however extremes, and

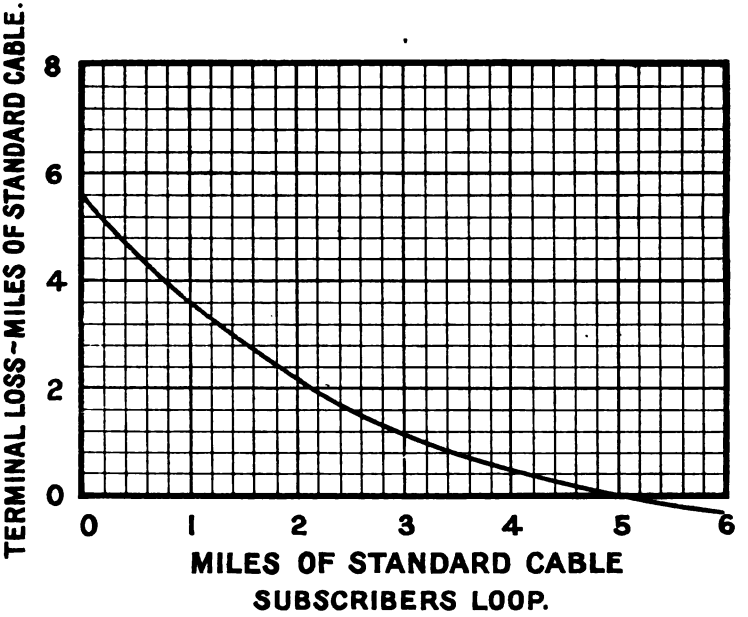

FIG. 147.—Terminal Loss at both ends of a Heavily Loaded Phantom Circuit using Standard Apparatus. Normal spacing (terminating half a loading section from each end).

there is evidently an intermediate stage between them. We thus are led to consider the desirability of different standards of transmission.

The General Transmission Scheme in the United Kingdom.—The United Kingdom is divided up into large areas known as "zones" for purposes of telephonic communication. Each zone has a telephone zone centre or centres through which communication with places in the same zone and in other zones is effected.

For very long-distance traffic certain selected zone centres serve as principal centres of communication, and it is between these selected zone centres that the main arteries of long-distance communication are provided. Such circuits are reserved for long-distance traffic where this is considered economical.

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In some cases where a second centre, known as a sub-zone centre, could be utilized owing to its favourable position with respect to

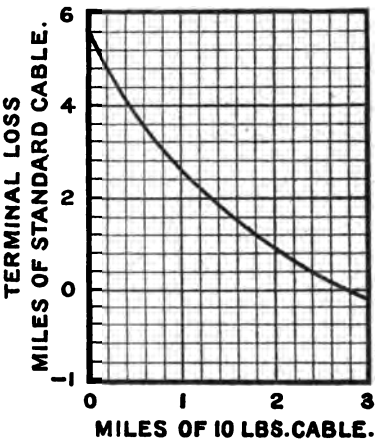

FIG. 148.—Terminal Loss at both ends of Heavily Loaded Phantom Circuit using Standard Apparatus and 10 lbs. Subscribers' Cable. Normal spacing (terminating half a loading section from each end).

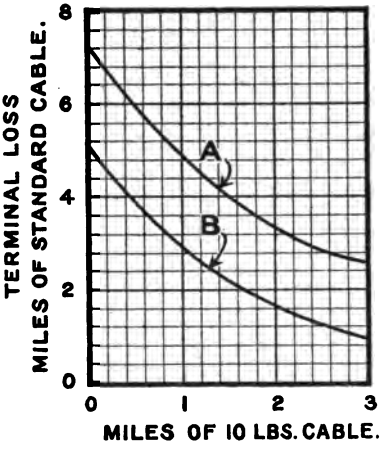

FIG. 149.—Total Transmission Loss at both ends of—

A. Heavily loaded aerial line terminating 1½ loading sections from each end.

B. Lightly loaded underground cable terminating 1 loading section from each end. Using Standard Apparatus on 10 lbs. Subscribers' Cable.

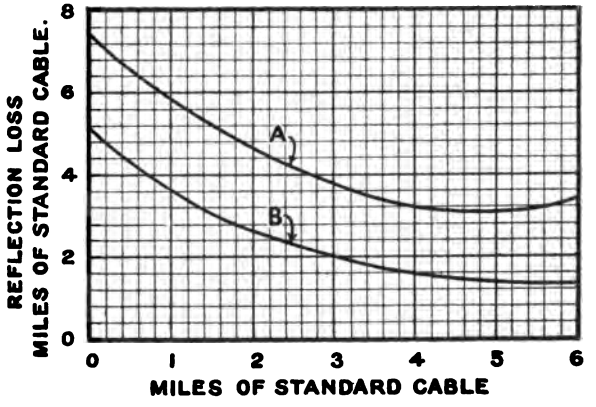

FIG. 150.

Same as Fig. 149, except that Standard Cable is used for the Subscribers' Line.

other zone centres, and to towns and localities in its own zone, such a sub-centre is provided with the object of shortening the lines of

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communication and facilitating telephonic traffic. The outline map (Fig. 151) shows the position of the zone and sub-zone centres in

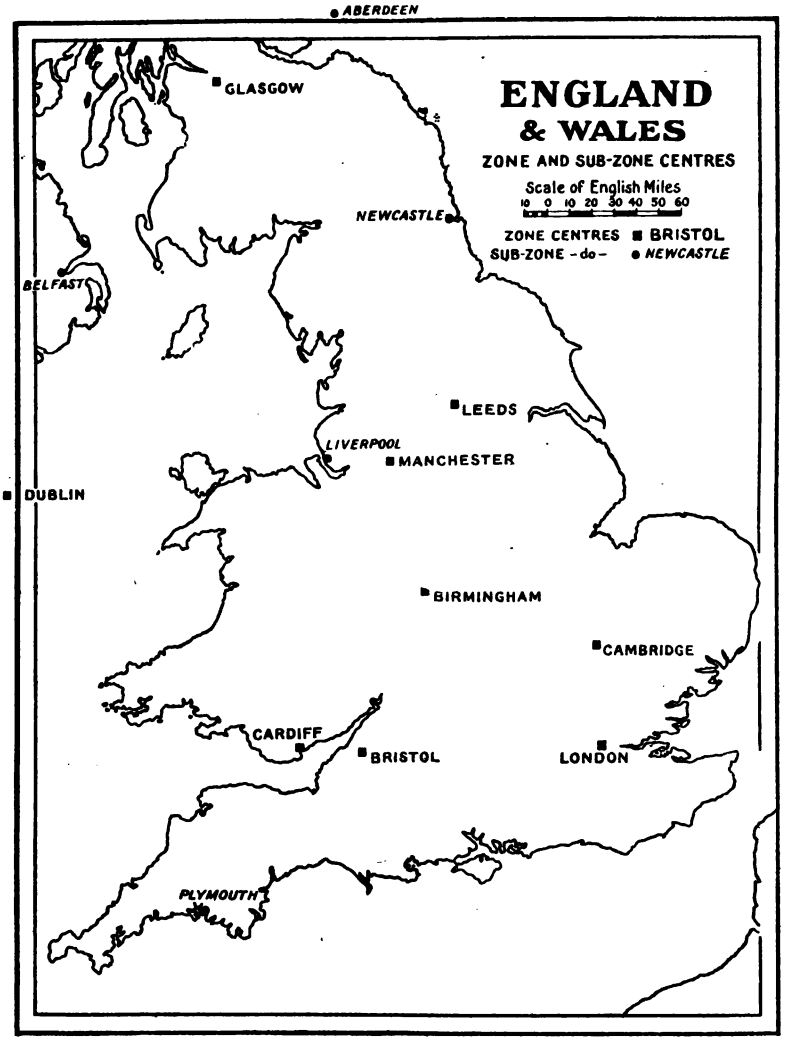

FIG. 151.—Map showing the Zone and Sub-Zone Centres in the United Kingdom.

England. Scotland and Ireland each constitute a separate zone with sub-zone centres at Aberdeen and Belfast respectively.

Fig. 152 has been prepared to give an idea of the method of

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providing intercommunication among the various groups in a zone, and its communication with other zones. In reality the system is much more complex than that indicated by the plan, but a relatively simple diagram is given to bring out the fundamental facts more clearly. The controlling exchanges correspond to the towns already mentioned as being the centre of a local system, and the minor exchanges are generally smaller towns, or villages, any of which might also if necessary communicate with other exchanges than those to which they are shown connected. The controlling exchanges control or regulate the traffic between the minor exchanges connected to them, and all other exchanges, *i.e.* they join the minor exchanges through to each other or to distant towns, in the order

EXPLANATORY DIAGRAM OF ZONE COMMUNICATION.

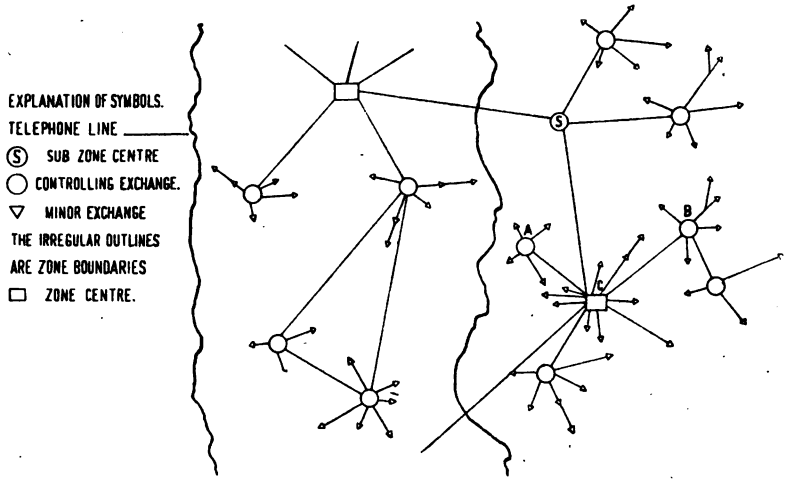

FIG. 152.

of priority of demand irrespective of the office of origin, and also supervise the duration of the calls. All calls passing through the zone centre are, however, controlled there. There is a notable exception to this method of control, *i.e.* if any two controlling or minor exchanges, each of which is connected to a third through which they communicate, have a considerable volume of traffic between them, arrangements are made where possible to organise the traffic on a no-delay basis, *i.e.* the controlling office in such a case does not note the time of demand and arrange the call in turn with others, but immediately joins the calling circuit direct to the required town. To illustrate this point, suppose we have intense traffic between A and B passing through C (Fig. 152), and that the necessary junction circuits to carry it exist between A and C and B and C. In the supposed case C would simply connect A with B

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immediately on the demand of A or B for as long as required, *if the circuits could be spared*, and would clear the connection on receiving a signal of termination.

This scheme gives practically unlimited use of circuits between A and B, when they are not required for use for traffic between A and C or B and C.

233. The Definition of Transmission Standards.—The following standards of telephonic transmission were laid down by the Engineer-in-Chief of the Post Office in May, 1915, under the heading of

"PREPARATION OF TRANSMISSION SCHEMES.

"The standards of transmission to be provided in future for local and trunk traffic will be as follows:—

"The audibility shall not be worse than that obtained from two standard telephones separated by the length of standard cable shown opposite the different classes of service detailed below.

Class of service.	Standard.
I. Between stations in the same urban area .	20 M.S.C.
II. Between (*a*) stations in the same zone. .	
Between (*b*) stations in adjacent zones and not more than 200 miles apart . . .	
Between (*c*) stations in any two large commercial centres where traffic is sufficient to justify direct lines between local exchanges; *e.g.*, London-Birmingham, London-Liverpool, Bristol-Cardiff . .	25 M.S.C.
III. Between (*a*) stations in adjacent zones and more than 200 miles apart; *e.g.*, Ipswich-Exeter	
Between (*b*) stations in different (not adjacent) zones and not more than 300 miles apart	30 M.S.C.
Between (*c*) stations (on central exchanges) in any two large commercial centres from 300 to 400 miles apart; *e.g.*, London-Dublin, London-Glasgow	
IV. General standard for long-distance communication between stations in Great Britain and Ireland and not included in any of the foregoing	35 M.S.C.

"In order that the standards quoted above may be attained, the standard cable equivalent of the combination of circuits by means of which a subscriber reaches his zone centre for a trunk call must not be greater than the equivalent of 10 miles of standard cable plus a 300-ohm loop, but the transmission efficiency of the combination of line circuits must in no case be inferior to that of an equal length of standard cable, that is to say, if the route length be three miles the standard cable equivalent of the line must not be more than three miles.

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"The standard cable equivalent of the junction circuit combinations used for trunk traffic between zone centres and the local exchanges of their immediate neighbourhoods must not exceed the following values :—

	Maximum standard cable equivalent.
Up to 10 miles route length	5
From 10 to 15 miles route length	6
,, 15 to 20 ,, ,, ...	7
,, 20 to 25 ,, ,, ...	8
Over 25 miles route length	10

"These standards include all wiring and apparatus losses."

234. The Design of a Transmission Scheme.—The economical lay-out of a telephone system to meet a fixed transmission standard presents a problem of some complexity. There are a number of possible solutions to the problem, which, however, cover a considerable range of cost. The object of the transmission engineer should be to obtain the required efficiency of transmission at the least possible cost. As a first step it is necessary to settle three important matters—

(*a*) To define the boundaries of the exchange area.

(*b*) To determine the theoretically correct position of the exchanges in the area, both as regards the centre of the whole area and each separate exchange.

(*c*) To fix transmission standards for local traffic and also for medium and long-distance communication.

The fixing of the boundaries will depend upon the community of interest of the prospective groups of subscribers, and also on the convenience of administration, bearing in mind that the whole area will require to be worked in conjunction with contiguous areas.

To find the ideal centre for a local exchange in a given town, a list of the known and probable subscribers is compiled. A large scale map of the town is then obtained and divided into squares of about 22,000 square yards in area. The position of each subscriber is marked by a dot in its respective square, and the total number of subscribers in each square is then entered in the square. All the columns of figures are next totalled horizontally and vertically. The sum of the horizontal and vertical columns will, of course, give twice the total number of subscribers. The totals of each horizontal column are entered opposite to and immediately at the end of the horizontal column to which they refer, and the vertical columns have their totals entered separately at the foot of each column.

We have thus two sets of totals, one vertical, consisting of the horizontal totals, and one horizontal, consisting of the vertical totals, the former being at the right-hand edge of the chart and the latter at its foot. Now find a point in each of these columns, such that the number of subscribers on both sides of it is equal. Thus if there are 50 totals in the vertical column, but the sum of the top 20 is equal to the sum of the bottom 30 totals, then the point required is at the foot of the 20th total from the top. Lines are drawn from each of the points named at right angles to the edge of the chart and to each other, and produced till they meet in the chart ;

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the point of junction is the theoretical centre, and the desirable or so-called ideal centre is the nearest possible point to it. (See a paper by R. Waring entitled "Development Studies," No. 23 in the Institution of Post Office Electrical Engineers, Professional Papers.)

The standard of transmission is determined by the general scheme already quoted.

As regards the question of deciding what proportion of the total transmission loss shall be allocated to any particular part of the total combination of circuits giving a fixed standard of transmission, one broad principle continually operates in transmission schemes. We have fundamentally to consider some local area consisting, let us say, of a large town and its suburbs, together with neighbouring smaller towns or villages near to it, all of these places probably having a community of interest. This local area, considered as a whole, will require to communicate with other areas, the nearer ones generally more frequently than the distant ones; and this localization of traffic will probably attain its maximum among the subscribers in the most important town of the local area. A number of junction circuits will be required for intercommunication among the various groups, and the number of these circuits will naturally depend on the extent of the traffic. The subscribers' circuits are evidently much more numerous than the junction circuits required for local traffic, and these are generally larger in number than the trunk circuits required for long-distance traffic. An important problem here at once presents itself, for if the subscribers in two towns in the same area are connected by a number of junction circuits and the whole local system is so designed that conversation between the subscribers is in accordance with some predetermined transmission standard, it is clear that the efficiency of the junction circuit and the subscribers' circuit may each vary considerably, and yet the combined efficiency of the two circuits may not exceed a given transmission standard. When, however, the cost is considered, there will probably be only one combination which gives the required efficiency at the cheapest cost, and this will generally depend on the relative amount of copper in the two groups of circuits. The problem outlined is a very frequently recurring one. In the next chapter some notes on the method of attaining the best results are given. It will be convenient at this point to anticipate this information to some extent.

Assuming that the positions of the exchanges have been determined and the length of the subscribers' lines is known, the standard can at once be applied.

Suppose we require to determine the circuits necessary for local intercommunication between two definite points in the area, it being known that the traffic will be sufficient to justify special circuits for local traffic, This case is illustrated by Fig. 153. After having thoroughly studied the method of determining the transmission equivalents in this case the same method may be applied to any other case. We have here two towns A and C (Fig. 153) communicating through a controlling exchange B, and we require to find the necessary lines and apparatus to meet the required local standard. The standard for local communication anywhere in the area is defined in the list of transmission standards already given.

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Conversation must never be worse than that obtained when speaking by means of the standard instruments over 20 miles of the standard cable. At the same time it must be remembered that the subscribers' circuit and apparatus must be suitable for long-distance communication.

At this point the specialized data furnished by the Engineer-in-Chief's office enables the engineer responsible for the work to arrive at a decision as to the type of circuits to be employed.

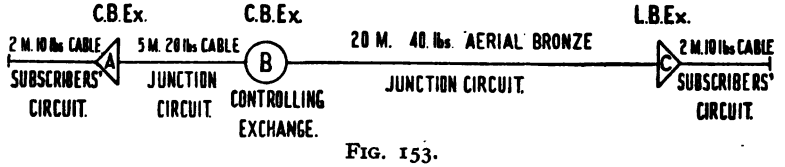

FIG. 153.

Selection of the Type of Office Equipment.—The type of telephone installation must now be decided; for example, the question arises as to whether a common battery or a local battery installation is to be installed. The present practice in Great Britain is to estimate on a twenty-year basis and to provide common battery installations where more than six operators' positions are required. The reasons for this do not come within the scope of this study. If the number of operators' positions will be not more than six a local battery installation is provided.

It is assumed in this particular case that common battery working will be required at A and B and a local battery system at C.

Selection of Conductors.—Transmission studies, such as those dealt with in the next chapter, show that it is economical to provide 10-lb. cables for subscribers in towns generally, and experience shows that by far the greater part of them come within a length of 2 miles. This maximum length and type of cable is assumed for the subscribers at A and C.

The subscribers' line being fixed, and the apparatus at the exchange known, the standard cable allowance permissible for the junctions may be found. It is, however, desirable in the first instance to express the standard of 20 miles of the test cable plus the standard instruments, in terms of the zero loop standard thus, remembering that the standard instruments contain a resistance of 300 ohms in both the sending and receiving circuits.

By Fig. 137 we have for a 300-ohm loop in the 24-volt system—

Sending allowance	10·2
Receiving allowance (experimental result) . . .	1·5
Local standard allowance	20·0
	31·7

This standard therefore must not be exceeded.

The losses due to the two 10-lb. subscribers' circuits, and the cord circuits, and office wiring at A, B, and C, are as follows. Assume in the first case that A is speaking to C, as this is the worst condition, as will be seen later. It is to be noted that if communication is not

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the same in the two directions, the worst case is always to be provided for. We here assume that the two most unfavourable subscribers' circuits have been selected. The losses involved are as follows:—

Subscriber at A, 2 miles of 10-lb. cable in C.B. 24-volt system. Sending allowance	11·5
Wiring allowance at A	0·4
,, ,, B	0·5
Cord loss at B	1·0
Wiring loss at C	0·4
Cord loss at C	1·0
Subscriber at C—2 miles of 10-lb. cable in local battery system—receiving allowance (by Table II., Chap. VIII.)	3·14
	17·94

(The subscribers' receiving apparatus is assumed to be equal to the zero loop standard.)

Deducting 18 m.s.c. from the available 31·7 m. standard we have 13·7 miles of standard cable for the junction circuits.

If the 20-mile junction is made up of 40 lbs. bronze, the transmission equivalent is 7 m.s.c. In view, however, of the fact that the characteristic impedance of this line is $1581 \angle 37° 46'$ ohms, the question of terminal loss requires examination. If the necessary information is not available from existing records it will be necessary to investigate the matter. If the bronze line be taken as sufficiently long to approximate to an electrically long line, and the apparatus and 2 miles subscribers' line at C has the same impedance as the standard apparatus on a similar local line, the transmission loss at C can readily be calculated. In any case, when the impedance is known the process of calculation is the same, and it will be convenient to assume that the impedance is that of the standard apparatus on a line with a resistance of 350 ohms or 4 miles of standard cable, as in Fig. 85, and that the bronze line is sufficiently long and has the impedance shown in Table II., Chap. VIII. Of course this assumption is not strictly correct, but the bronze line is extended by two others before reaching the subscriber at A. Mr. A. J. Aldridge, in his paper on "Practical Application of Telephone Transmission Calculations," referred to in connection with Table II., found that in the case of composite lines a constant value is reached when the lines have values approximating to 8 m.s.c. This is not far from the 7 m.s.c. equivalent of the bronze line in question. If some such assumption is not made the calculation becomes very complex. It *can*, however, be dealt with by constructing the equivalent π or T circuits, or by graphical methods such as that used by Mr. Aldridge. On the assumption we have made, however, the loss at C is found by example number 1 to be 0·5 m.s.c. The loss at the joint at B on making the same assumptions as to sufficient length is by example number 2 1·15 m.s.c. It is to be noted that any error resulting from the assumption that the lines are long enough to have a constant impedance is likely to be that a loss somewhat too great will be obtained. This is because the effect of electrically short lines

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inserted between long lengths of other lines, as illustrated by Fig. 84, is to decrease the loss, until, when the line is very short, the loss disappears altogether. It should be noticed that the ratio of the impedances of the bronze circuit and the standard cable is 2·77 with an angle of only 5½ degrees. The effect of the angle is here very small and the result obtained in example 1 is nearly the same as that obtained from Fig. 81, *i.e.* the case where the angle is neglected altogether; this is generally true when the difference of the angles involves a small figure.

From the preceding remarks it will be seen that the transmission loss at the junctions B and C amounts to 1·15 + 0·5 = 1·65 m.s.c. To arrive at this loss we have in the case of each end of the line assumed the bronze length to be inserted between two equal impedances and have divided the resulting loss by 2 to obtain the mean in both directions. The 20 miles junction circuit with its terminal loss will therefore account for the total of 7 + 1·65 = 8·65 m.s.c., leaving 13·7 − 8·65 = 5 m.s.c. nearly, available for the 5-mile junction circuit between A and B. If underground work is suitable as assumed, then 20-lb. cable would suffice, if the capacity of this cable were 0·054 mf. per mile, as it would have a standard cable equivalent of 5 m.s.c. If the capacity were higher this would be increased accordingly, and the cable would not meet the case. If, however, 40 lbs. bronze aerial wire were more economical and an open route suitable it would be used. In this latter case there would be ample margin of transmission efficiency, and the question of terminal loss would not be important. Its amount, however, would be easily found, for we have then a sufficiently long length of 40 lbs. bronze between two equal 10-lb. circuits, and this has already been dealt with.

In dealing with cord and wiring losses the remarks under the heading of "Applications of the Standard Cable Method of Testing," should be borne in mind.

Notes on the Standard Cable Values found in the preceding Example.—The sending allowance due to the 2 miles length of 10-lb. cable is found by taking the receiving allowance of a 350-ohm loop on the 24-volt system from Fig. 138 and adding it to the current supply loss for the same resistance in Fig. 139, in order to illustrate this method.

It will be observed that nothing is shown under the heading of cord loss at A; this is because the cord circuit is included in the standard test instrument.

No terminal loss is involved at Exchange A because the standard cord circuit with a 350 subscribers' circuit is connected to a length of standard cable.

Modification of Allowance when sending from C to A.—Assuming that cords of similar impedance are used in both directions the sum of the transmission values between A and C will be unaltered, but the subscribers' circuit values will be modified as shown hereafter. It should be noted, however, that the outgoing exchange cord has sometimes a different value to the incoming one, and this should be watched. As this does not affect the principle of the example we assume the same value in both directions. Suppose that an ordinary micro-telephone is used at C. This instrument has an

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allowance of 7·5 m.s.c. as compared with zero loop. The transmission losses in the subscribers' circuits are as follows:—

	Miles of Standard Cable.
Subscribers' apparatus at C, sending allowance (as compared with zero loop standard)	7·5
Subscribers' line at C, 2 miles of 10-lb. cable . . .	3·14
Receiving allowance of subscribers' line at A by Fig. 138	3·0
	13·64

A reference to the previous list of losses will show that those due to the subscribers' circuits when sending from A to C amount to 14·64 m.s.c., and it is therefore evident that the loss is greatest from A to C. This condition should therefore be provided for.

This difference of transmission in two directions does not conflict with the conclusions in Chap. XIII. as to communication being equal in two directions in stipulated terminal conditions, because in the case of Chap. XIII. equal voltages are applied to the ends of the line, whereas owing to the effect of the resistance of the common battery circuit on the battery supply, the effect in sending and receiving is different in the present case; and moreover the apparatus at the two ends is not alike, so that different voltages are applied at the two ends of the circuit.

Junction Circuits for Trunk Traffic.—If the junction circuits are to be used for trunk traffic as well as local traffic they must be of such an efficiency as to provide for the more exacting requirement, *i.e.* long-distance communication, and this is a very common case in small exchanges. If there is one group of circuits set apart for incoming circuits and another for outgoing circuits, then all the long-distance work will be carried over the outgoing junctions; and these must be of higher efficiency than the incoming junctions.

235. Transmission Losses.—Generally speaking such losses may be divided into line losses, apparatus losses and wiring losses in offices. It is a fundamental condition for finding the losses by calculation that the impedance of the lines and apparatus involved should be known.

Line Losses (i.e. *Terminal and Transition Losses*).—If the lines may be assumed to be sufficiently long, formulæ for the calculations will be found in Chap. VII. The graphs of terminal losses and transition losses in Chap. VII. and the present chapter furnish a large amount of information on this point. If calculations are required for short lines the data in Chap. VII. show how to calculate these results if Fig. 84, Chap. VII., does not apply.

Losses due to Apparatus bridged across Lines.—If the line is long enough such losses may be calculated from (34), etc., Chap. XIII. Such losses are practically the same at the sending and receiving ends of a line. For lines of high impedance, however (heavily loaded lines), it will be found that the losses due to bridged apparatus are greater at the centre of the line than at the ends, and may amount to double the loss at the ends; they are greater in loaded than in unloaded lines. For low resistance circuits, *i.e.* for most unloaded lines the losses are greater at the ends than the middle. These facts are deduced from the study of the formulæ in Chap. XIII.

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Losses due to Transformers and Condensers in Series in Lines.—As a rule these losses are obtained experimentally. The losses due to condensers in unloaded lines are nearly the same at the ends and the centre. In heavily loaded lines they are greater in the middle than at the ends of the circuit.

Combined Losses due to Apparatus in Bridge and in Series in Transmission Lines.—It has been shown by Messrs. K. Ogawa and T. Arakawa, in their article on "Telephonic Transmission Losses," in the *Telephone Engineer* for December, 1917, that if the impedance of the apparatus in series is relatively so small and the impedance in bridge is so large that the apparatus loss is small compared with the attenuation loss in the lines, then the sum of the separate losses will represent with sufficient accuracy the combined loss. The authors referred to find that this rule is of very general application.

Office Wiring.—Wires of a given gauge in silk and cotton covering or in silk and wool covering will generally have a transmission loss about one and a half times that of wires of similar weight as given in Table II., Chap. VIII. The average loss for a small exchange is 0·4 m.s.c.; for a medium sized exchange 0·6 m.s.c.; and for a large exchange 1 m.s.c.

The following list of transmission losses is in use in the Engineering Department of the Post Office. The varying resistances given are, of course, for identification purposes, and do not give a reliable clue to the transmission losses. It may be mentioned that the current circulating in electromagnetic apparatus may have a material effect on transmission losses.

The following table gives the equivalents, in terms of standard cable, of various apparatus when joined up in shunt across telephone circuits :—

TABLE I.

TRANSMISSION EQUIVALENTS OF APPARATUS.

Type of apparatus.	Transmission equivalent M.S.C.	Remarks.
Bell magneto, 1000ω	0·5	
,, 1000ω in series, with 2 mf. condenser	0·5	
Coil bridging, 600ω	1·0	
,, 1000ω	0·5	
,, 600 + 600ω	0·5	
Transformer 7 terminal, when used as bridging coil	0·5	
Coil retardation, 600ω	1·0	
,, 1000ω	0·5	
Indicators (tubular type), 1000ω	0·5	
Relays, 1000ω	0·5	
Coils, repeating, 37A	2·0	
,, ,, 4005A	1·0	These values are the losses due to transformation.
,, ,, 4006A	0·75	
Transformer No 1	1·0	
,, 7 terminal	6·0	

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TRANSMISSION EQUIVALENTS OF CORD CIRCUITS AND EXCHANGE APPARATUS.

Diagram No.	Description of circuit.	Transmission equivalent M.S.C.
TL 26 . .	Lamp Signalling Trunk Exchanges Cord Circuit	0·5
,, 60 . .	Trunk Switch Section D Cord Circuit . . .	0·5
,, 510 . .	,, ,, ,, DI ,, ,, . . .	0·5
,, 559, Fig. 2	Non-Lamp signalling Trunk Exchanges Cord Circuit associated with Jack-ended Junctions (where lines were previously worked as trunks).	0·5
,, 561, Figs. 1 and 2	Terminal equipment for Jack-ended Junctions at Trunk Exchanges	1·0
,, 560 . .	Trunk Exchange Lamp Signalling Jack-ended Junction working, Cord Circuit for . . .	0·5
,, 562 . .	Lamp Signalling Exchanges, Ringing Junctions, Terminal Equipment	1·0
,, 565 . .	Trunks terminated at Magneto Exchanges, Terminal Equipment	1·0

The following table of impedance of apparatus is taken from Mr. A. J. Aldridge's "Applications of Telephone Transmission Calculations," read before the Institution of Electrical Engineers on May 30th, 1913. It relates to items extensively employed in the Post Office Telephone Service, and may be applied to obtain useful results from the formula in Chap. XIII.

The following details of impedance are taken from the records of the Post Office Engineering Department:—

	Load on Secondary. Ohms.	Current.	Impedance. Modulus.	Angle.
Transformer 4006A . .	800	0·5 milliamp	945	/8° 35′
	600		756	/7° 58′
	1000		1139	/9° 27′
Telephone for Local Line, P.O. type		0·3 milliamp	340	/47° 15′
Telephone for Local Line, N.T. Co.'s type			678	/63° 30′

The impedance was measured at $2\pi f = 5000$.

236. Transmission Records.—Records under the following heads are desirable in every telephone area. A complete survey is not necessary, but the information should be recorded and kept up-to-date. The information available should include in all cases the mileage, details of line conductors, and apparatus and wiring in exchanges, together with the highest standard cable value in any group of circuits.

A convenient arrangement of details is as follows:—

(1) Subscriber connected to each exchange (including private branch exchanges, etc.) sending and receiving allowances.

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TABLE II.—IMPEDANCE OF APPARATUS AT TELEPHONIC FREQUENCIES.

Description of apparatus.	Frequency.	Current (milliampères).		With conditions of Columns 2, 3, and 4.			Variation of impedance likely to occur in practice.		Remarks.
		Continuous.	Alternating.	Resistance.	Inductance.	Impedance.	At 800 ~.	At all frequencies.	
				ohms.	henrys.	ohms.			
60ω Bell D.P. Receiver. Used on most C.B. instruments and on P.O. L.B. instruments.	800	0	0·3	134	0·030	207/50	200/50 to 224/48	130/45 to 310/52	Very little affected by change of continuous or alternating current.
120ω H.M.T. Receiver. Used on most N.T. Co. L.B. instruments.	800	0	0·3	268	0·058	397/47	390/47 to 450/46	260/44 to 650/53	Ditto.
500ω tubular retardation coil. Iron jacketed about 9 × 2 cm. Used as a bridging coil.	800	0	0·3	3,526	0·602	4,652/40·5	3250/44 to 9930/41	2670/48 to 14,890/38	Impedance decreases with increase of continuous current.
600ω tubular retardation coil. Iron jacketed about 6 × 2·5 cm. Used as a bridging coil.	800	0	0·3	4,880	0·912	6,690/34·2	5220/46 to 10,600/43	3520/47 to 12,400/45	Impedance decreases with increase of continuous current.
160ω retardation coil, closed magnetic circuit, double coil, wire core, each coil 5 cm. diam., 5 cm. long. Used as bridging coil on private branch exchanges.	800	0	0·3	1,790	5·266	26,480/86	8850/88 to 27,000/88	4000/87 to 50,000/81	Impedance decreases with increase of continuous current.
200ω Knife-edge relay, copper jacketed. Used as bridging relay.	800	0	0·3	2,048	0·462	3,100/49	3000/50 to 4800/47	2000/50 to 6300/39	Very little affected by change of continuous current.
1000ω magneto bell ..	800	0	about 0·3	10,660	2·550	16,430/50	—	—	—
1000ω indicator used on magneto circuits, and 1000ω iron jacketed retardation coil. Used as bridging coil.	800	0	about 0·3	16,000	1·170	17,070/20	—	—	—
200ω tubular retardation coil. Used in series with similar relay as bridging coil on Peel Conner circuits.	800	0	0·3	1,624	0·300	2,185/42	1700/48 to 3000/42	1000/37 to 4000/43	—

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(2) Local junctions (incoming, outgoing, and combined).
(3) Trunk junctions.
(4) Cord losses and wiring losses at exchanges.
(5) Terminal losses.
(6) Trunk circuits.
(7) Losses due to apparatus in bridge or in series other than cord circuits.

With this information available it is possible to find the standard cable equivalent of any combination of circuits having the greatest transmission loss in any direction by simple addition of the standard cable values.

The late National Telephone Co. made a large number of transmission records of the areas in the United Kingdom, and did much valuable work in this connection. This information is still in existence and is used by the officers of the Post Office.

237. Use of the Minus Sign to denote Gain in Transmission.—Cases may arise where a transmitter or receiver gives louder transmission than the standard apparatus on zero loop. By comparison with the standard apparatus this represents a relative transmission gain instead of a loss. Suppose that we have to deal with such a case. It is evident from the example dealt with that all the losses are added together to ascertain whether the combination comes within the standard allowance. So long as there are nothing but losses the question of prefixing the plus or minus sign to the figures does not enter into the matter, but if losses and gains come into the algebraic sum then we are called upon to decide whether the losses shall be called +, as was the frequent practice of the Post Office, or −, as was the practice of the National Telephone Co. In view of the fact that the National Telephone Co. had by far the largest number of local systems, and naturally also of records referring to them, it is clear that considerable confusion is avoided by keeping to the National Telephone Co.'s practice, so far as transmission schemes are concerned and in many other cases.

238. Importance of the Maintenance of Transmission Standards.—Before concluding this chapter attention is called to the importance of maintaining a standard of transmission when once it is laid down. The fixing of a standard involves the supposition that the cost of providing it has been fully considered on the one hand, and balanced against the benefits to the department and the public on the other hand, the conclusion being that the maintenance of the standard is in the best interests of all concerned; and it should not therefore be departed from except for very serious reasons. If, for example, a reasonable standard were degraded the inevitable result would be complaints and loss of traffic and probably in the long run nothing would be gained. On the other hand, if conductors are provided giving better transmission than the transmission requirements, unnecessary expense is involved. It must, however, be admitted that cases may arise where the expense of bringing very unfavourably situated localities within the general standard would be prohibitive; such cases should only form a small proportion of the whole if the standard is to be worthy of its name.

A problem which frequently arises is as to whether for local convenience or some reason which in itself may be desirable, it is

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permissible to slightly reduce a standard of transmission between two given points. It can generally be shown that if the proposed reduction (even half a mile of standard cable) had been permissible much more economy could have been effected in using thinner conductors in the first instance than is likely to result from any present economy. Further, if any such small deterioration is allowed, it is not likely to be limited to one case, and the accumulation of such cases would have a very serious effect on the transmission of the country in the long run.

(For examples on Telephonic Development, see Chap. XII.)

Example 1.—Find the terminal loss between a subscribers' C.B. circuit of 350 ohms resistance in 10-lb. cable and a long length of 40 lbs. bronze.

In this case it will be convenient to find the loss with similar apparatus to that described at both ends of a long length of 40 lbs. bronze circuit. The relative formula is No. (60), Chap. VII., or

$$A = \tfrac{1}{2} + \tfrac{1}{4}\left(\frac{Z_0}{Za} + \frac{Za}{Z_0}\right) = \tfrac{1}{2} + \tfrac{1}{4}\left(\frac{1581\,\angle 37^\circ\,45'}{838\,\angle 12^\circ\,20'} + \frac{838\,\angle 12^\circ\,20'}{1581\,\angle 37^\circ\,45'}\right)$$

The impedance of 40 lbs. bronze is taken from Table II., Chap. VIII., and of the 2 miles subscribers' line from records. The impedance taken in this case is that of a 10 lbs. line of 350 ohms, or a line 2 miles in length. Continuing the calculation, we have

$$\begin{aligned} A &= \tfrac{1}{2} + \tfrac{1}{4}(1{\cdot}887\,\angle 50^\circ\,5' + 0{\cdot}530\,\angle 50^\circ\,5') \\ &= \tfrac{1}{2} + \tfrac{1}{4}(1{\cdot}210 - j\,1{\cdot}447 + 0{\cdot}340 + j\,0{\cdot}407) \\ &= \tfrac{1}{2} + \tfrac{1}{4}(1{\cdot}550 - j\,1{\cdot}040) = \tfrac{1}{2} + 0{\cdot}387 - j\,0{\cdot}260 \\ &= 0{\cdot}887 - j\,0{\cdot}260 = 0{\cdot}9243\,\angle \end{aligned}$$

This is the vector whose $\log_e$ gives the required equivalent βl or

$$\log_e 0{\cdot}9243 = -0{\cdot}0787$$

Expressing this in terms of miles of standard cable, we have

$$\frac{-0{\cdot}0787}{0{\cdot}106} = -0{\cdot}742$$

This is an apparent *gain* of nearly $\frac{3}{4}$ m.s.c., but by comparison with the standard cable when connected to the standard instrument it is a *loss*. Since the latter, by Fig. 82, on a 4-mile length has a gain of 1·78 m.s.c. The loss is therefore $1{\cdot}78 - 0{\cdot}742 = 1{\cdot}038$ m.s.c.

This is the loss at both ends of the 40 lbs. bronze line, the average loss in both directions (sending and receiving) for one end is

$$\frac{1{\cdot}038}{2} = \underline{0{\cdot}5 \text{ m.s.c.}}$$

Example 2.—Find the transition loss between the standard cable and a 40 lbs. bronze circuit, both lines being assumed to be long.

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By Table II., Chap. VIII., we obtained the required impedances whence $A = \frac{1}{2} + \frac{1}{4}\left(\frac{Z_0}{Za} + \frac{Za}{Z_0}\right)$ as in Example 1, or

$$A = \tfrac{1}{2} + \tfrac{1}{4}\left(\frac{1581\angle 37^\circ\ 45'}{571\angle 43^\circ\ 16'} + \frac{571\angle 43^\circ\ 16'}{1581\angle 37^\circ\ 45'}\right)$$

and proceeding as in Example 1 we find that $\frac{\log_e A}{0{\cdot}106} = 2{\cdot}3$.

As before the loss for one end of the line is half or $\frac{2{\cdot}3}{2} = 1{\cdot}15$ m.s.c.

This is a transition loss only and the comparison with the standard does not enter into it.

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CHAPTER XII

SOME COST PROBLEMS IN TELEPHONIC TRANSMISSION

239. The Minimum Cost of a Loaded Underground Cable, including Coils, for a given Transmission Efficiency or Equated Length.—*Statement of the Problem.*—The cost of a long underground cable may amount to a sum in five figures expressed in pounds sterling. Assume that the circuits in the cable are required to have some known attenuation constant. It is known that this constant may be obtained by means of conductors of very different weight per mile, when combined with suitable values of inductance: thus if a cable with an attenuation constant of 0·016 is required, this could be provided by stock types of cable conductors consisting of 40 lbs., 70 lbs., or 100 lbs. per mile, combined with standard loading coils providing 210, 52, and 24 millihenrys per mile respectively, but the relative cost of coils and cable and the total cost of a cable pair loaded with the necessary inductance per mile would be very different in the three cases. An important factor in the choice to be made is the relative price of the cable pairs and loading coils. There is some combination of cost of a cable pair and its loading which constitutes a minimum. It may happen that a study of costs based on the available stock items alone gives a price far from the minimum, and it is evidently desirable to know the extent of the difference, so as to determine whether any large saving would justify a new stock item, especially if a similar requirement were likely to recur.

By means of the following formula, due to the author, a price P is obtained for any combination of conductor resistance and inductance per mile, which will give the required attenuation constant. The constants A, B, and C_1, as defined hereafter, must be known, and from them x_2 is found. Both the required resistance R_2 and inductance L_2 are functions of x_2, and they can be readily obtained from it.

The proof involves considerations of some complexity; but the application of the formula is not difficult, as shown by the example at the end of the chapter. In view of the high cost and possible economy to be obtained by studies of this kind, they are of course highly desirable. The formula is

$$P = \frac{A}{C_1}\left(x_2 + 2 + \frac{1}{x_2}\right) + \frac{B\sqrt{C_1}}{(x_2 + 1)} \quad . \quad . \quad . \quad (1)$$

All the symbols in the equation relate to one mile of cable conductor pair. The resistance R and inductance L of one pair,

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and the resistance R_2 and inductance L_2 of a second pair, referred to hereafter, are resistances and inductances which give in every case the same β.

P = the combined cost of one cable pair and its loading.

A = RP_1, where R is the resistance of a known and selected conductor pair and P_1 its cost.

B = $\sqrt{\frac{x}{R}}P_2$, where $x = \frac{L_m}{L}$. L_m is the maximum possible loading of the conductor of resistance R, and L is the necessary loading which, combined with R, gives β. P_2 is the total cost per mile of providing L, and includes coils and manholes.

$C_1 = R\left(x + 2 + \frac{1}{x}\right) = R_2\left(x_2 + 2 + \frac{1}{x_2}\right)$. Here $x_2 = \frac{L_{m2}}{L_2}$, which is the maximum loading L_{m2} for R_2 divided by an inductance L_2, which inductance, when associated with R_2 in a cable pair, gives the required β. R_2 and L_2 are the unknown values to be found from x_2 from equation (1).

x_2 is defined under C_1.

240. Proof of Equation (1).—(*a*) *The relation between the inductance per mile and the distance apart and cost of the coils.*—From (17), Chapter IX.,

$$CDL_c = 25 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If L is the inductance per mile of a cable pair, neglecting its small natural inductance, we have

$$\frac{L_c}{D} = L \quad \text{or} \quad L_c = LD \quad . \quad . \quad . \quad . \quad (3)$$

Insert this value of L_c in (2) and we obtain

$$CLD^2 = 25 \quad \text{or} \quad LD^2 = \frac{25}{C} \quad \text{and} \quad \sqrt{L}D = \sqrt{\frac{25}{C}} \quad . \quad (4)$$

It is assumed in this case that C is a constant and therefore the product $\sqrt{L}D$ in (4) is a constant and equal to $\sqrt{\frac{25}{C}}$.

If L in such a case is multiplied and divided by *a*, (4) becomes

$$\sqrt{aL}\frac{D}{\sqrt{a}} = \sqrt{\frac{25}{C}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From this it is evident that if L is increased *a* times the distance D between the coils is reduced $\sqrt{a}$ times, since $\sqrt{\frac{25}{C}}$ is constant.

Example.—(*a*) If the inductance per mile L is quadrupled, the distance between the coils must be halved if nothing else changes. It follows that the price in this case is doubled, since the number of loading coils and loading points is doubled.

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It follows from (4) that the cost of loading varies as the square root of the inductance per mile.

Now by (4), Chap. IX.,

$$L_m = \frac{R}{\frac{R_1}{L_1} + \frac{G}{C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and as shown in the definition of B

$$x = \frac{L_m}{L} \quad \text{or} \quad L = \frac{L_m}{x}$$

and therefore by (6) $$\frac{L_m}{x} = \frac{R}{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)x} = L$$

or

$$\sqrt{\frac{L_m}{x}} = \sqrt{\frac{R}{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)x}} = \sqrt{L} \quad . \quad . \quad . \quad (7)$$

Similarly for any other conductor R_2, with a maximum inductance L_{m_2}, we have

$$\sqrt{\frac{L_{m_2}}{x_2}} = \sqrt{\frac{R_2}{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)x_2}} = \sqrt{L_2} \quad . \quad . \quad (8)$$

Dividing (7) by (8) we have

$$\sqrt{\frac{L_m}{x} \times \frac{x_2}{L_{m_2}}} = \sqrt{\frac{R}{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)x} \times \frac{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)x_2}{R_2}} = \sqrt{\frac{L}{L_2}} = \frac{P_2}{P_3} \quad (9)$$

The relation of cost $\frac{P_2}{P_3}$, where P_3 is the cost of L_2, follows from the fact that the price varies as the square root of the inductance. Since P_2, the price of L, is by supposition already known, then from (9)

$$\text{Cost per mile of } L_2 = P_3 = P_2\sqrt{\frac{R_2}{R} \times \frac{x}{x_2}} \quad . \quad . \quad . \quad (10)$$

but $$P_2\sqrt{\frac{x}{R}} = B \text{ and therefore } P_3 = B\sqrt{\frac{R_2}{x_2}} \quad . \quad . \quad (11)$$

(*b*) *Cost of a Pair of Conductors in an Underground Cable.*—The prices of the copper conductors, whose resistances are R and R_2, are assumed to vary inversely as their resistances. If R is the resistance per mile loop of a cable pair which costs £P_1 per mile, then the cost per mile of the cable pair of resistance R_2 is $\frac{R}{R_2}P_1$. In a cable made up of pairs all of the same weight, this price is here

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taken as the total cost of the cable divided by the number of pairs in it. If the price of R_2 is called P_4, then

$$\frac{P_4}{P_1}=\frac{R}{R_2} \quad \text{or} \quad P_4=\frac{R}{R_2}P_1=\frac{W_2}{W}P_1 \quad . \quad . \quad (12)$$

where W and W_2 are the weights per mile of copper in the conductors of resistance R and R_2 respectively.

(*c*) *The Relation between R, L, and x, and R_2, L_2, and x_2.*—From App. VII. we have

$$R\left(x+2+\frac{1}{x}\right)=R_2\left(x_2+2+\frac{1}{x_2}\right) \quad . \quad . \quad (13)$$

where $x=\frac{L_m}{L}$ and $x_2=\frac{L_{m2}}{L_2}$. From this it is seen that L is a function of x, and from (13) we have

$$R_2=\frac{R\left(x+2+\frac{1}{x}\right)}{\left(x_2+2+\frac{1}{x_2}\right)}=\frac{C_1}{\left(x_2+2+\frac{1}{x_2}\right)} \quad . \quad . \quad (14)$$

from the definition of C_1 already given.

(*d*) *Formation of Equation for the Total Price of a Loaded Cable Pair.*—The total price, P, of a loaded cable pair of resistance R_2 and inductance L_2 having a fixed attenuation β, is the sum of P_3 and P_4, and by (11) and (12) this is

$$P=\frac{RP_1}{R_2}+B\sqrt{\frac{R_2}{x_2}} \quad . \quad . \quad . \quad . \quad (15)$$

but by definition $RP_1=A$, and from (14) $R_2=\frac{C_1}{x_2+2+\frac{1}{x_2}}$.

Substituting these values of RP_1 and R_2 in (15) we obtain

$$\begin{aligned}P&=\frac{A}{C_1}\left(x_2+2+\frac{1}{x_2}\right)+\frac{B}{\sqrt{x_2}}\sqrt{\frac{C_1}{\left(x_2+2+\frac{1}{x_2}\right)}}\\&=\frac{A}{C_1}\left(x_2+2+\frac{1}{x_2}\right)+\frac{B\sqrt{C_1}}{\sqrt{x_2}\left(\sqrt{x_2}+\frac{1}{\sqrt{x_2}}\right)}\\&=\frac{A}{C_1}\left(x_2+2+\frac{1}{x_2}\right)+\frac{B\sqrt{C_1}}{(x_2+1)} \quad . \quad . \quad . \quad . \quad . \quad (16)\end{aligned}$$

We have in (16) an equation with only one unknown quantity, x_2. By means of this equation the minimum cost of a cable pair in combination with loading coils giving a fixed attenuation constant β_1 may be found in two ways :—

(1) By means of a graph showing the relation between x_2 and the total cost P, as shown in Fig. 155 (see Example 1). From this curve the minimum price P, and the corresponding value of x_2, are found, and R_2 and L_2 then deduced from (14) and (8).

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(2) By differentiating (16) and finding the minimum value of x_2, and then substituting this value of x_2 in (16).

Using method (2) we have by differentiation of (16)

$$\frac{dP}{dx_2} = \frac{A}{C_1}\left(1 - \frac{1}{x_2^2}\right) + B\sqrt{C_1} \times \frac{-1}{(x_2+1)^2}$$

$$= \frac{A}{C_1}\left(1 - \frac{1}{x_2^2}\right) - \frac{B\sqrt{C_1}}{(x_2+1)^2} \quad . \quad . \quad . \quad . \quad (17)$$

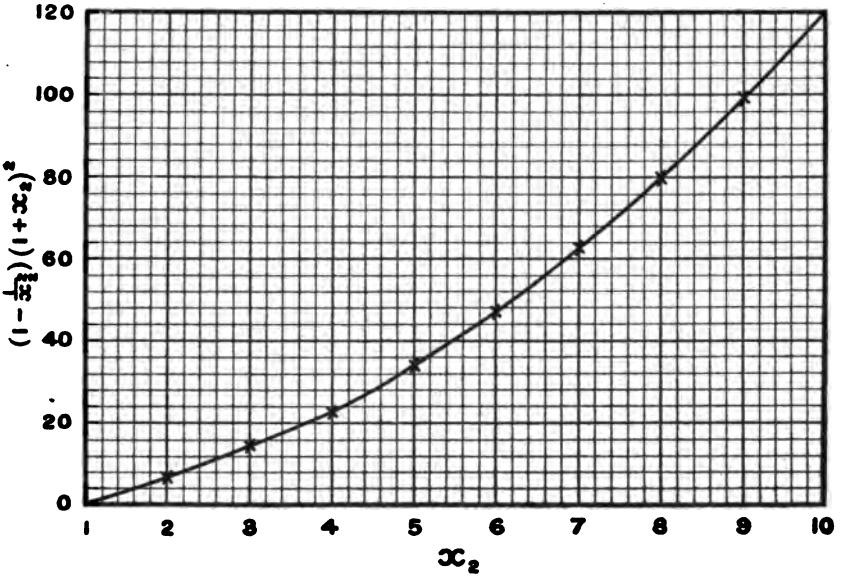

FIG. 154.

For a maximum or minimum value $\frac{dP}{dx_2} = 0$, whence

$$\frac{A}{C_1}\left(1 - \frac{1}{x_2^2}\right) = \frac{B\sqrt{C_1}}{(x_2+1)^2} \quad . \quad . \quad . \quad . \quad (18)$$

or

$$\left(1 - \frac{1}{x_2^2}\right)(x_2+1)^2 = \frac{BC_1\sqrt{C_1}}{A} \quad . \quad . \quad . \quad (19)$$

This gives a minimum value.

From this equation x_2 can be found when the value of $\frac{BC_1\sqrt{C_1}}{A}$ has been computed from known constants. In order to facilitate the solution of (19) the value of $\left(1 - \frac{1}{x_2^2}\right)(x_2+1)^2$ has been computed for all values of x_2 between 1 and 10. The relation is shown by Fig. 154.

(*e*) *Considerations affecting the Phantom Circuit.*—When the

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weight of conductor of the side circuit and the spacing distance of that circuit are fixed, there is only a limited choice as to the resulting efficiency of the phantom circuit. In such a case all that can be done to obtain the best result is to consider the efficiency of the loading coil to be used in relation to its cost, and to load as heavily as the distance apart of the manholes permits, having due regard to the spacing rule (C.D.$L_e = 25$ in this case). It appears at first sight as though the transmission efficiency of the phantom circuit would be materially greater if heavy gauge conductors are used as compared with the results obtained with lighter ones. This, however, does not necessarily follow unless we have freedom of choice as to the spacing distance apart of the manholes, and such is usually not the case. The effect on the phantom circuit of a given loading of the side circuit should, however, be carefully considered. If full advantage is taken of the spacing rule to place coils in manholes as wide apart as possible, it will be found that in air-space underground cables side circuits require *closer* spacing than phantom circuits to produce the same attenuation constant. As a matter of fact, the phantom circuits in the London-Liverpool cable (see Chap. IX.) are about 15 per cent. more efficient for transmission with the same spacing distance than the side circuits on which they are superimposed. The figures given are, however, by no means rigid, and the improvement of 15 per cent. may be exceeded. If, therefore, the side circuit is designed for a given efficiency, we know that the phantom circuit will, generally speaking, be somewhat better. The method here adopted is therefore to aim at a definite efficiency for the side circuit without specifically mentioning in the calculation the efficiency of the phantom circuit. The full *cost* of loading that circuit is, however, taken into account, and the efficiency of the phantom circuit is largely determined by the loading distance of the side circuit (see para. 239).

(*f*) *Note on the Assumptions made as to Costs of Coils and Conductors—Loading Coils.*—If the loaded circuits are all of the same gauge and are not phantomed the matter is relatively simple, and the cost follows the rule given without qualification. If, however, the circuits are all superimposed, then, as a general rule, these circuits have not the same inductance as the side circuits, if they are calculated for the same efficiency. For costing purposes we have to consider that, as a rule, only one set of manholes will be used, and that it is the coils which require the closest spacing which fix the cost. When the spacing distance is fixed, the plan here adopted is to assume a certain necessary cost for laying the cable in the absence of the coils, and to add to this the cost of the coils and any extra charges, such as manholes for the coils and charges for their installation. These added items are included in the cost of loading. When once the cost for providing, accommodating, and fixing one loading point is calculated, the cost per circuit per manhole is this cost divided by the number of coils at one point. If the circuits are phantomed the cost taken is that for a phantom unit or three circuits. Knowing the number of loading points and the length of the cable, the cost per mile is then easily found. It is assumed that the cost of laying the cable will be the same whatever gauge of conductor is selected.

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Conductors.—The supposition that the cable space required is approximately in proportion to the relative weight of the conductors is based on the practice of the British Post Office, and the thickness of the lead sheath is also taken as being approximately a simple function of the diameter of the core for sufficiently large cables. For a capacity of 0·065 mf. per mile wire to wire, the spaces allowed per pair for multiple twin conductors suitable for superimposing (20 lbs. per mile and over) are given in para. 153, Chap. VIII.

241. The Minimum Cost of a Composite Loaded Underground Cable, including Phantom Coils, the Cable having a Given Length and a Fixed Equated Length.—If a loaded cable contains two or more types of conductors the use of one set of loading points for all types of conductors simplifies the maintenance of the cable and permits of considerable economy in the construction of manholes for the coils, even when the ideal loading points for the different types of conductors employed are not the same. The calculation of the minimum cost is, however, more complicated in the case where only one set of loading points is available. If each set of conductors of different types were unrestricted as regards loading points, the method of calculation explained in para. 240 could, in fact, be employed. It may be remarked that the restriction of loading points, which follows from the fact that if we are confined to one set of manholes we can only make the theoretically best arrangement for one type of conductor, renders it impossible to obtain the minimum theoretical cost for more than one set of circuits in the same cable at one time; and, moreover, it does not follow that this minimum, when taken in connection with the costs for the other types of conductor in the same cable, will give the best total value.

The minimum cost for a cable containing two types of conductors, both fully loaded for phantom working, will now be considered, but the principle outlined will apply to a larger number of circuit types with only one set of loading points. The following is a method of finding the minimum cost :—

(*a*) Find the required attenuation constants, as explained in Example 1, and then find the values of A, B, and C_1. The following notes may be of assistance in calculating A and B. The value of C_1 apparently presents no difficulty. The cost of a pair in this case may be determined as follows, bearing in mind the limitations as to accuracy already laid down. Let the weight of one mile of single wire of one type be called a and the weight of a mile of the other type be called b, also let the total weight of a length of one mile of all the combined conductors in the cable be called c; and, finally, let the total price of one mile of the cable be called d pounds sterling. Then the price per mile of two pairs required for a phantom unit is

$$\left.\begin{aligned}\text{Cost of two pairs of type } a &= \frac{4ad}{c}\\ \text{Cost of two pairs of type } b &= \frac{4bd}{c}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The resistance of the selected conductor *pair* per mile multiplied

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by twice the cost of a pair per mile gives the required constant

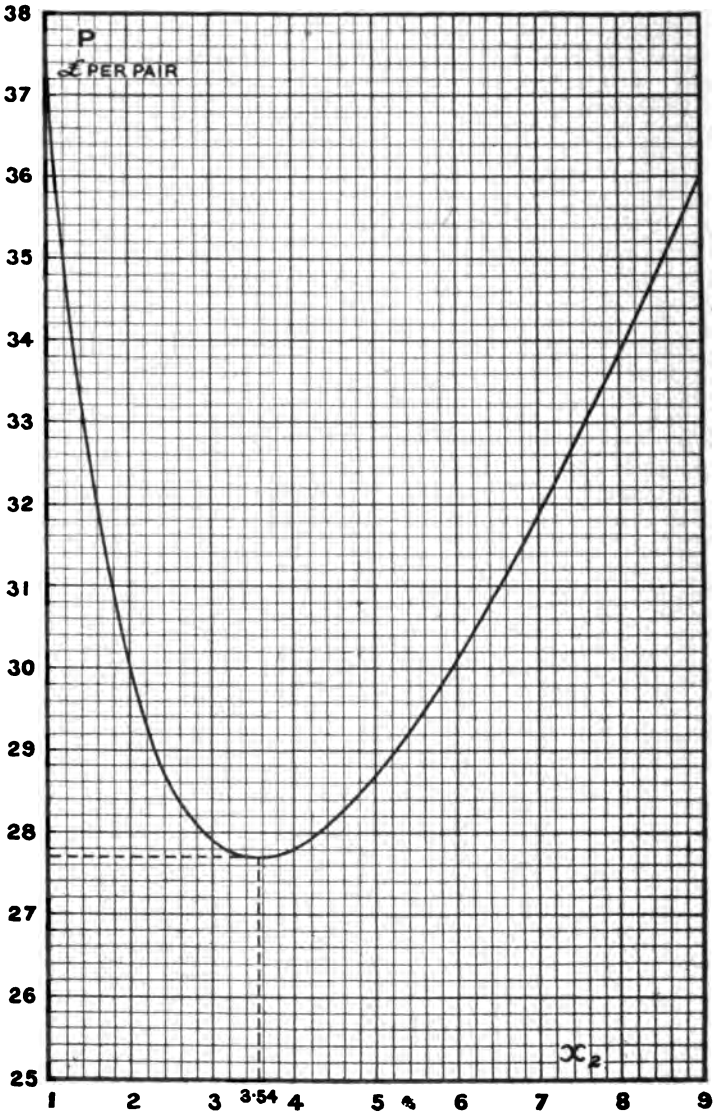

FIG. 155.—Relation between the Degree of Loading x_2 and the Total Cost of 1 Mile of Loaded Circuit.

A in each case. If the same coil is used for each type of side circuit and if the coils to be used for the two types of phantom circuit are

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alike, then the cost for loading will be alike for both types of circuits. In any case, the cost per mile for loading at some given spacing distance must be known for each loading unit made up of two sides and one phantom circuit. When these data are known, B can be calculated as in Example 1.

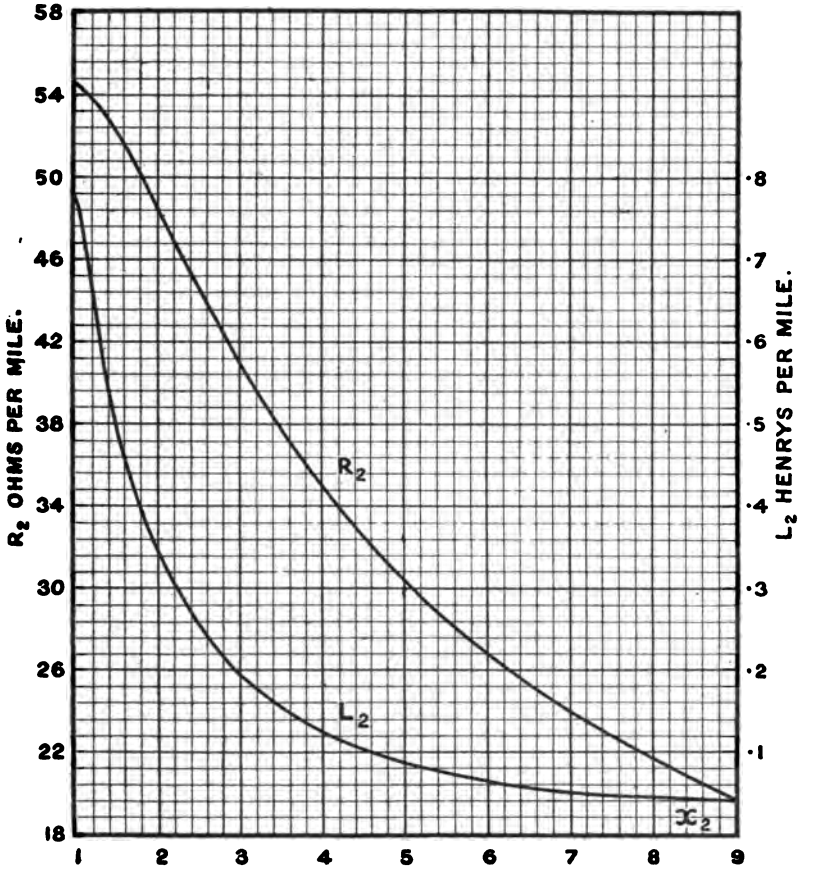

FIG. 156.—Graph showing the Combined Resistances and Inductances which give, in every case, the same Attenuation Constant 0·016.

(*b*) After finding A, B, C_1, and x_2, construct graphs similar to Figs. 155 and 156.

(*c*) Next construct two curves on one and the same graph, showing the relation between the price per loaded pair multiplied by the number of pairs of each type, and the inductance of the pairs of that type corresponding to the price P and P_s. Thus, if there are n pairs of one type and m pairs of the other type, and

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the price of one pair is P as determined by equation (1), and the price of the other type is P_s as determined also by equation (1), then the total price of all conductors of the one type is nP and of the other type mP$_s$. The corresponding inductance of a pair whose price is P or P_s is found from graphs such as Figs. 155 and 156, which are supposed to be available, and L_2 is then plotted against the price nP, and on the same graph another curve connecting L_2 and nP$_s$ is constructed.

(*d*) Find the least sum of the ordinates of these two curves for any one given value of L_2; this is the minimum cost of the cable.

To prove that the minimum cost of the cable is obtained with circuits of different types having the same inductance L_2, where this is possible, and when only one set of loading points is used, we have, by paras. 239 and 240,

$$C_1 = R_2\left(x_2 + 2 + \frac{1}{x_2}\right)$$

It is evident from this equation that as x_2 is made smaller (and consequently L_2 is increased) R_2 is increased (see (14)), and therefore the weight and cost of the conductor are reduced. Now, if any selection whatever of two values of inductance for the two sets of circuits is made from the graph for the loading of the two types of conductors in question, and one value of L_2 is higher than the other, this fixes the loading point, which must be that for the higher inductance in order that *both* circuits loaded by these inductances fulfil the spacing rule (17), Chap. IX. Now, by hypothesis the circuit having the lower inductance must be fixed at the same point as the other, since there is only one set of manholes, and therefore its inductance may be increased without cost until it is equal to the other. It then follows, as pointed out in connection with the preceding equation, that the conductor resistance must be increased to keep the attenuation constant of the same value as it was before L was increased, unless we have maximum loading. The circuit as now constituted will, in fact, be the cheapest possible, since the weight of copper is reduced to a minimum and the cost of loading cannot be further reduced or altered. It follows that the combination of two circuits which have the same inductance, and at the same time indicate the minimum sum of the prices in the curve, cannot be further reduced in price, and therefore give the required minimum cost.

The phantom circuit attenuation constant should be taken into account in making a final choice. The importance to be attached to it depends largely upon the requirements of the service for which it is provided and the cost of the coils.

In preparing the curve, A, B, and C_1 can be varied in the case of larger and smaller conductors if greater accuracy is required.

242. (*a*) **For a fixed attenuation length (βl) in aerial line telephonic transmission, a conductor of uniform gauge throughout necessitates the least weight of copper.** (*b*) **If any section of the line has a different weight per mile as compared with the other section, the total combined copper weight will always be less the nearer the copper weights per mile are to equality.**—As regards

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(*a*), strictly speaking this proof applies to copper conductors of 200 lbs. per mile and upwards, but there is only a small error in assuming that all copper conductors used for aerial line telephony in Great Britain obey the law (the least gauge used is 100 lbs. per mile per wire).

Let it be assumed that formula (47), Chap. VIII., applies to any uniform aerial conductor of resistance R, length l, and weight W per mile. This will be sufficiently correct for our object, with the limitations stipulated in the last paragraph.

Next let it be assumed that the same total attenuation as given by the conductor R of length l can be obtained by means of two different lengths l_1 and l_2 (where $l_1 + l_2 = l$) of conductors having resistances R_1 and R_2 and weights per mile per wire W_1 and W_2 respectively, one of these conductors being necessarily heavier and the other lighter than the conductor, whose weight is W lbs. per mile. It is assumed that the inductance, capacity and leakance are equal in all cases. The sum of the attenuation constants multiplied by the respective lengths in such a case is therefore assumed to be as follows :—

$$\frac{l\left(R+\frac{G}{C}L\right)}{2}\sqrt{\frac{C}{L}}=\frac{l_1\left(R_1+\frac{G}{C}L\right)}{2}\sqrt{\frac{C}{L}}+\frac{l_2\left(R_2+\frac{G}{C}L\right)}{2}\sqrt{\frac{C}{L}} \quad (21)$$

Eliminating the common factor $\frac{1}{2}\sqrt{\frac{C}{L}}$ and multiplying by the various lengths we have

$$Rl+\frac{G}{C}Ll=R_1l_1+\frac{G}{C}Ll_1+R_2l_2+\frac{G}{C}Ll_2$$

or
$$Rl-R_1l_1-R_2l_2=\frac{G}{C}Ll_1+\frac{G}{C}Ll_2-\frac{G}{C}Ll$$

that is
$$Rl-R_1l_1-R_2l_2=\frac{G}{C}L(l_1+l_2-l)$$

and since $l_1 + l_2 = l$ we have

$$Rl-R_1l_1-R_2l_2=0$$

or
$$Rl=R_1l_1+R_2l_2 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

When these relations hold, the attenuation length βl of the circuit of resistance R and length l is equal to the combined attenuation lengths of the circuits having the resistances R_1l_1, and R_2l_2; but the combined weight of the latter conductors may be very different to that of Rl. This may be verified by the following formulæ :—

By (2), Chap. VIII., the weight varies inversely as the resistance, and if A be a constant, the following relations are evident. We have

$$W=\frac{A}{R} \quad W_1=\frac{A}{R_1} \quad W_2=\frac{A}{R_2} \quad . \quad . \quad . \quad (23)$$

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If $\frac{A}{R_1}$ be multiplied by l_1 and $\frac{A}{R_2}$ be multiplied by l_2, the total weight W_3 becomes

$$W_3 = \frac{Al_1}{R_1} + \frac{Al_2}{R_2} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

To obtain the most economical circuit for a fixed transmission value in this case, the sum of $\frac{Al_1}{R_1} + \frac{Al_2}{R_2}$ must have a minimum value, whilst at the same time the sum of $R_1l_1 + R_2l_2$ in equation (22) remains constant. This may be brought about as follows.

If in equation (22) R_1l_1 is increased by a value x so that x is uniformly distributed over the conductor R_1 for the whole length l_1, then, in order that the total value of the equation may remain constant, R_2l_2 must be decreased by the amount x equally distributed over l_2. In that case (22) becomes

$$Rl = (R_1l_1 + x) + (R_2l_2 - x) \quad . \quad . \quad . \quad (25)$$

(24) then becomes

$$W_3 = \frac{Al_1}{R_1 + \frac{x}{l_1}} + \frac{Al_2}{R_2 - \frac{x}{l_2}} \quad . \quad . \quad . \quad . \quad (26)$$

(x is divided by l_1 and l_2 to reduce it to a value per mile.)

The minimum value of this expression, when W_3 is differentiated with respect to x, is obtained as follows:—

$$\frac{dW_3}{dx} = -\frac{Al_1^2}{(R_1l_1 + x)^2} + \frac{Al_2^2}{(R_2l_2 - x)^2}$$

For a maximum or minimum value we have

$$\frac{l_1^2}{(R_1l_1 + x)^2} = \frac{l_2^2}{(R_2l_2 - x)^2} \quad \text{whence} \quad \frac{l_1}{l_2} = \frac{R_1l_1 + x}{R_2l_2 - x} \quad (27)$$

If x be written $r_1l_1 = r_2l_2$, where r_1 and r_2 are the respective resistances added to R_1 and subtracted from R_2, (27) then becomes

$$\frac{l_1}{l_2} = \frac{R_1l_1 + r_1l_1}{R_2l_2 - r_2l_2} = \frac{l_1}{l_2} \times \frac{R_1 + r_1}{R_2 - r_2}$$

whence

$$R_1 + r_1 = R_2 - r_2 \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Now $R_1 + r_1$ is the adjusted resistance of R_1, and $R_2 - r_2$ is the adjusted resistance of R_2, and when finally adjusted they are equal and uniform, as shown by (28), which proves the proposition so far as the uniformity of the conductors is concerned.

A simple application of (22) is of interest. Suppose that $l_1 = l_2$, then, since $l_1 + l_2 = l$, both l_1 and l_2 become $\frac{l}{2}$. Formula (22) then becomes

$$Rl = R_1\frac{l}{2} + R_2\frac{l}{2} \quad \text{or} \quad R = \frac{R_1 + R_2}{2} \quad . \quad . \quad (29)$$

These are values per unit length, in this case per mile.

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It follows from the operations leading to (22) that if the attenuation constants of the circuits R, R_1, and R_2 are substituted in formula (29) for the corresponding resistances they will have the same relation to each other as the resistances; thus if R is the resistance of a 300 lbs. circuit, R_1 that of a 600 lbs. circuit, and R_2 that of a 200 lbs. circuit, we have by Table II., Chap. VIII., $R = 5{\cdot}87$, $R_1 = 2{\cdot}93$, $R_2 = 8{\cdot}80$, and (29) becomes by substitution of these values $5{\cdot}87 = \frac{2{\cdot}93 + 8{\cdot}80}{2}$, which is easily seen to be correct. The corresponding attenuation constants by the same table are, for 300 lbs. Cu 0·00488, for 600 lbs. Cu 0·00279, and for 200 lbs. Cu 0·0069. These values should, by the deduction now made, have the relation

$$0{\cdot}00488 = \frac{0{\cdot}00279 + 0{\cdot}0069}{2} \quad . \quad . \quad . \quad (30)$$

This is very nearly correct, as the actual figure obtained from the right-hand side of (30) is 0·00485. The largest error in the use of the formula is when dealing with the smaller gauges in the aerial line table referred to. For example, taking the data of 300 lbs. copper and 100 lbs. copper conductors to find the attenuation constant of a 150 lbs. copper circuit, it will be found that the error is a little less than 3 per cent.; for the heavier conductor circuits it is much less.

(*b*) This may be easily proved as follows for the case where $l_1 = l_2$ as given by equation (29). If the resistances R_1 and R_2 are varied so that R_1 increases to $R_1 + r$ and R_2 decreases to $R_2 - r$, the total value of (29) is unaltered, but the corresponding weight of copper is then by modification of (23), if W_4 is written for $W_1 + W_2$

$$W_4 = (W_1 + W_2) = \frac{Al_1}{R_1 + r} + \frac{Al_1}{R_2 - r} \quad . \quad (31)$$

or

$$W_4 = l_1\left(\frac{A}{R_1 + r} + \frac{A}{R_2 - r}\right) \quad . \quad . \quad . \quad (32)$$

or

$$W_4 = l_1\left\{\frac{A(R_2 - r) + A(R_1 + r)}{(R_1 + r)(R_2 - r)}\right\} = \frac{l_1 A(R_1 + R_2)}{R_1R_2 + r(R_2 - R_1 - r)} \quad (33)$$

Assume that the only variable is r, then the only part of (33) containing the variable is $r(R_2 - R_1 - r)$. As this quantity is in the denominator of (33), the weight W_4 will be a minimum when it, the variable part, is a maximum. Differentiate the variable part of (33), thus

$$y = r(R_2 - R_1 - r) \quad . \quad . \quad . \quad . \quad . \quad (34)$$

$$\frac{dy}{dr} = R_2 - R_1 - 2r \quad . \quad . \quad . \quad . \quad . \quad (35)$$

For a maximum or minimum value we have

$$2r = R_2 - R_1 \text{ or } r = \frac{R_2 - R_1}{2} \quad . \quad . \quad . \quad (36)$$

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If, however, (35) is differentiated it gives $\frac{d^2y}{dr^2} = -2$, which is a negative value, and therefore it leads to a maximum value. It follows that when this condition is attained the weight equation (33) will have a minimum value. We are now in a position to prove (*b*). To do this, first express y in terms of its maximum value by substituting the value of r from (36) in (34), thus

$$r(R_2 - R_1 - r) = \frac{R_2 - R_1}{2}\left\{R_2 - R_1 - \frac{(R_2 - R_1)}{2}\right\} = \left(\frac{R_2 - R_1}{2}\right)^2 = r^2 \quad (37)$$

Next substitute the value of $R_2 - R_1$ in (34) by its equivalent $2r$ from (36), and write (34) as follows:—

$$r_1(R_2 - R_1 - r_1) = r_1(2r - r_1) = 2rr_1 - r_1^2 \quad . \quad (38)$$

Now (37), as already proved, is the maximum value of (34), and (38) is a variable expression for all other values of (34). Since r_1 is the only variable, subtract (38) from (37) and we obtain

$$r^2 - 2rr_1 + r_1^2 = (r - r_1)^2 \quad . \quad . \quad . \quad . \quad (39)$$

This equation shows at a glance the difference between the maximum value of (34) (which determines the minimum weight in (33)) and any other value of (34); the greater the difference the greater will be (33), the total weight of the wire; this proves (*b*) for wires of equal length. The same facts may be proved by the graphical method.

Messrs. A. B. Hart and W. J. Hilyer proved in their paper on Transmission Schemes, which is referred to later in this chapter, that the foregoing law holds as regards small gauge cable circuits, in which case the attenuation constant is

$$\beta = \sqrt{\tfrac{1}{2}\omega CR}$$

It will therefore be seen that the law which states that a uniform gauge of circuit for a given attenuation length involves the minimum weight of copper is of wide application, and it is scarcely necessary to add that it is of the highest importance that it should be observed wherever possible.

248. Cost Curves for Local Telephone Systems—Notes on the Costs of Telephone Systems.—Reference is made in para. 234, Chap. XI., to the frequently recurring problem in transmission which presents itself when provision is made for connecting a group of circuits of a given number and of known lengths with another group of a different number and length. As an example we will consider the problem of connecting a large number of subscribers' circuits in a given local area with a given number of junction circuits of known length, the transmission not to exceed the limits of a definite known standard; say, for example, 10 miles of standard cable when utilizing the standard instruments (Fig. 135, Chap. XI.). The problem consists in finding a method of ensuring that the standard shall never be exceeded and *at a minimum cost*. Suppose, as a starting point, that the longest subscribers' circuit is assumed to be provided by means of 20 lbs. cable, and that the

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junction circuits are so provided as to then give the required standard. It is evident that all subscribers' lines of materially less length than the longest one could be provided by means of thinner copper than 20 lbs. cable. Again, if the junction circuits were more efficient than just supposed, the gauge of the 20 lbs. longest subscribers' circuit could be reduced and all the others reduced proportionately. The problem then resolves itself into distributing the copper among the subscribers' and junction circuits in such a way as to fulfil the standard conditions at a minimum cost.

So far as large exchange areas are concerned in this country, the degree of development is such that it is hardly likely that the problem of providing an entirely new network on a very large scale—say for 10,000 subscribers—will present itself. The problem, however, frequently presents itself for solution for smaller areas and for opening new exchanges and providing for extensions of existing ones. The principle is always the same, and the method here adopted is of general application. The first solution to be considered is a graphical one.

Suppose that a preliminary survey has been made, as described in Chap. XI., and that the position of the exchanges, with the number and length and route of the lines, is known. The cost of providing the subscribers' lines to a given standard of transmission may be arrived at as follows. First construct a graph, such as Fig. 157, to show the percentage of all the subscribers' lines connected to the given exchange, which are not longer than some specified length. This length is given for all lengths of subscribers' lines up to 2 miles radius in the supposed case, and it is assumed that 99 per cent. of the subscribers are included in it. Thus 70 per cent. of the subscribers have lines not longer than 0·75 mile, and 10 per cent. are not more than 0·15 mile in length.

Fig. 158 gives the aggregate length of all the lines connected to the exchange, which are not longer than some length stated in the abscissa of the graph. This curve is based on the preceding one and is calculated from it, but it could be made up from available figures in an actual case. For example, the graph shows that 3,230 miles of circuit in the aggregate is the length of all the lines not more than one mile in length, and 260 miles is the total length of all lines not more than 0·25 mile in length.

Applications of the Curves.—We will assume that it is required to find the best distribution of copper among the 10,000 subscribers in a common battery system and an appropriate number of junction circuits of known length connected to known exchanges. As a result of past experience assume to begin with that 6½ lbs. and 10 lbs. cables will be used, and that the resistance of the subscribers' lines will not exceed 350 ohms on a standard common battery system. The 6½ lbs. cables are to be used wherever possible, and the 10 lbs. cables where absolutely necessary. Theoretically a weight of copper less than 6½ lbs. per mile could evidently be used for the very short lines, but no such cable is at present available. From the graphs we know the number of subscribers and the lengths of their lines at any point, and these may be localized if required from the available maps to definite cable routes. From

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these data an average price per mile of subscribers' circuit may be found, based on previous departmental experience in similar areas. Seeing that the number of circuits at every distance is known, it will be possible to find approximately the cost of distri-

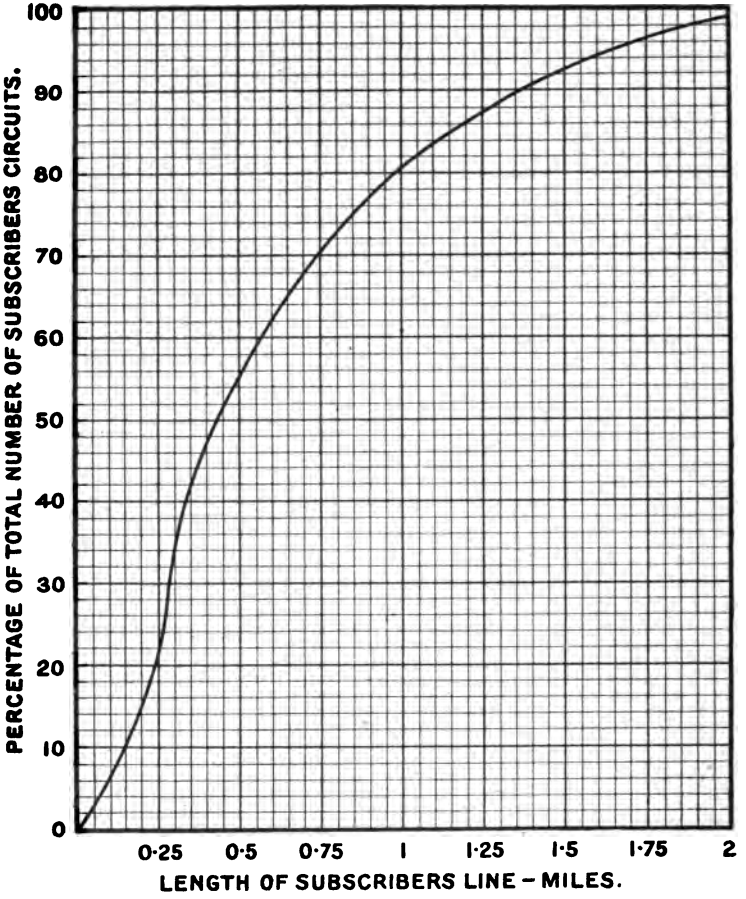

FIG. 157.—Graph showing what percentage of the Subscribers connected to a given Telephone Exchange have Circuits which do not exceed a Given Length.

bution from the cable ends and of leading in to the subscribers' offices and providing apparatus.

The transmission efficiency absorbed by the most unfavourable subscribers' circuit can be found, and the wiring and cord losses in all the exchanges affected arrived at, as the details are supposed to be known. These separate amounts for the worst connection

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in the area must next be totalled, and this amount deducted from the permissible total allowance or standard, *i.e.* 10 miles of standard

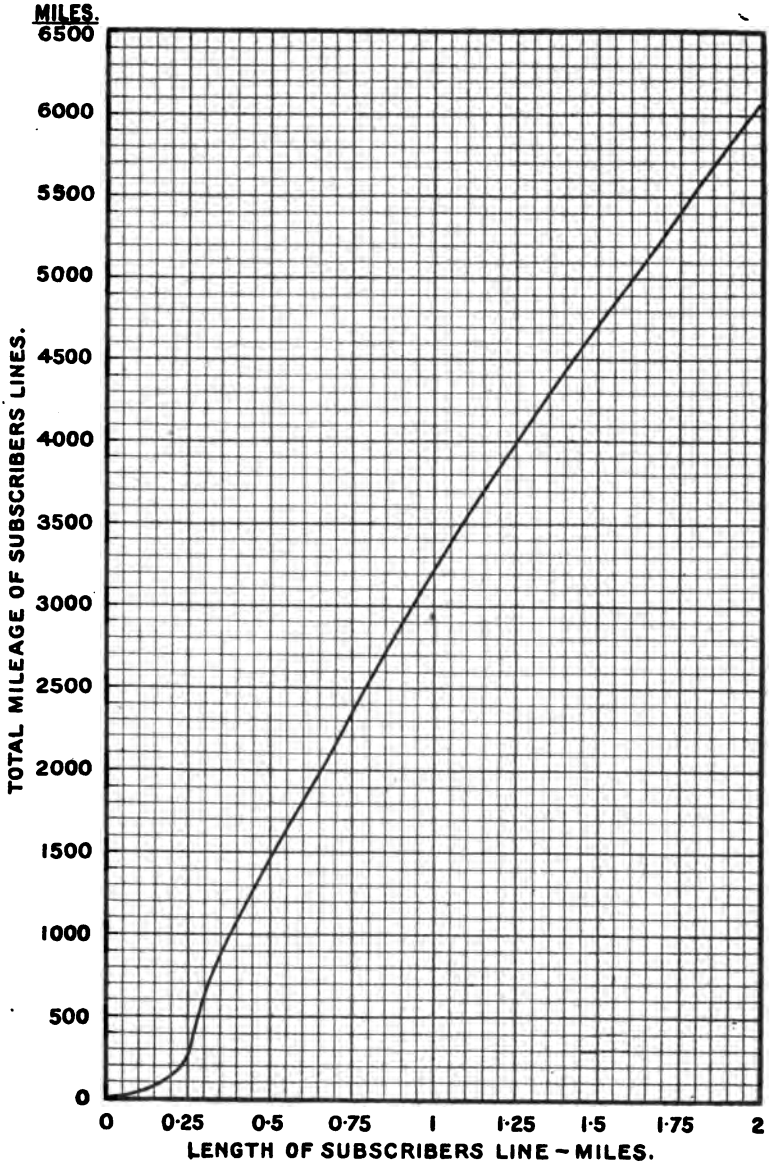

FIG. 158.—Graph showing the Total Mileage of Subscribers' Line in an Area having any Radius up to 2 Miles from the Exchange.

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cable and the standard instruments, or 21·7 miles of standard cable on zero loop standard (10 m.s.c. standard + 10·2 miles standard cable for the sending allowance + 1·5 m.s.c. for the receiving allowance). The difference so arrived at is available for the junction circuits, and it fixes the type of circuit which must be utilized for them within definite limits, and therefore regulates their cost.

Now, if we have freedom of choice as to the amount of resistance which may be allowed for the subscribers' circuits, then so long as the total permissible transmission equivalent is not exceeded we may assume that some other resistance is allowed as a maximum instead of 350 ohms—say, for example, 300 ohms. Acting as before and selecting cables of such a weight as to give the required resistance only, it will naturally be found that the weight of conductor used in the subscribers' circuits is greater than before, and the cost of the subscribers' circuits greater. The cost of the junction circuits will, however, be smaller, in view of the fact that the subscribers' 300 ohms circuits will be more efficient than in the previous case, and therefore a larger standard cable equivalent will be available for the junction circuits, which may therefore be of smaller gauge. It can now be seen whether the sum of the cost of the subscribers' and junction circuits is greater in the former or the latter case. Proceeding in this way with other maxima for subscribers' circuit resistances, the cheapest combined cost may be found. The more extensive and complicated the junction scheme, the more it becomes economical to reduce the resistance of the subscribers' circuits, and vice versa. The subscribers' circuits are, however, also required for trunk communication, and the cost of providing this with various resistances in the subscribers' lines must be borne in mind. In the particular instance we have in view, the 10-mile standard aimed at allows for trunk communication.

Such studies involve a considerable amount of work, but, on the other hand, the cost of providing the system is very great. A solution for such a case can in some cases be found by an application of the calculus, as shown later; but it must again be borne in mind that a graphical method shows at a glance what is involved in departing from the minimum cost, which is a great advantage in many cases. With the calculus method only the minimum point is known, and many approximations are necessary in cases of this kind. It is, however, frequently convenient and sufficient in the case of preliminary estimates to utilize a formula to determine the minimum cost.

244. Minimum Cost Formulæ.—The following formula has been devised to determine the minimum cost in cases where the transmission law assumed in the formulæ holds good. This formula and formula (59) which follows it, were developed and given by Messrs. A. B. Hart and W. J. Hilyer of the Engineer-in-Chief's Office, Post Office, in a paper read on March 10th, 1913, before the Institute of Post Office Engineers, and entitled, "Trunk Telephone Communication Transmission Schemes and the Design of Circuits." See the *Electrician*, April 11th, 1913. The formula can be applied to any two groups of circuits. For example, in Fig. 159, if a total standard cable equivalent E is permissible for

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communication between 1 and 3, and there are N circuits L miles long between 1 and 2, and n circuits l miles long between 2 and 3, the formula (No. (49)) gives the minimum cost for the necessary combined circuits between 1 and 3. Exchange No. 1 may be a zone centre or any other exchange; Exchange No. 2 may be a junction centre or any other exchange; and No. 3 may be an exchange or a subscribers' station. It will be convenient for our purpose to prove the formula for the latter case.

Let L = length of a junction circuit.
,, l = ,, ,, subscribers' circuit.
,, N = number of junction circuits.
,, n = ,, ,, subscribers' circuits.
,, E = the equated length (m.s.c.) allowed for the junction circuit and subscribers' circuit combined.
,, x^2 = the weight of Cu in a subscribers' line 1 mile in length.
,, y^2 = ,, ,, ,, junction circuit ,, ,,

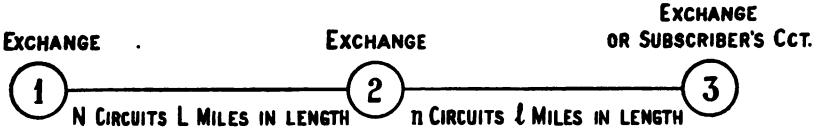

FIG. 159.

The formula for β for a light gauge underground cable is $\beta = \sqrt{\frac{1}{2}\omega CR}$ by (57), Chap. VIII. If $\omega = 5000$ and C = 0·054 mf., this becomes

$$\beta = 0{\cdot}0116\sqrt{R} \text{ per mile loop} \quad . \quad . \quad . \quad (40)$$

But $R = \frac{1719}{W}$ per conductor pair, where W is the weight of wire per mile per wire. (See (15), Chap. VIII.)

$$\therefore \beta = 0{\cdot}0116\sqrt{\frac{1719}{W}} = \frac{0{\cdot}481}{\sqrt{W}} \quad . \quad . \quad . \quad (41)$$

Therefore the standard cable equivalent of 1 mile of cable circuit of weight W is equal to

$$\frac{0{\cdot}481}{0{\cdot}106} \times \frac{1}{\sqrt{W}} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

From the above definitions we have the total weight of copper W_1

$$W_1 = NLy^2 + nlx^2 \quad . \quad . \quad . \quad . \quad . \quad (43)$$

Seeing that $W = x^2$ for 1 mile of subscribers' lines and $W = y^2$ for 1 mile of junction circuit, (42) may be modified and written $\frac{0{\cdot}481}{0{\cdot}106}\left(\frac{1}{y} + \frac{1}{x}\right)$, and for the total lengths of circuit L and l

$$E = \frac{0{\cdot}481}{0{\cdot}106}\left(\frac{L}{y} + \frac{l}{x}\right) \quad \text{or} \quad \frac{0{\cdot}106E}{0{\cdot}481} = \frac{L}{y} + \frac{l}{x} \quad . \quad (44)$$

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In this case $\frac{0{\cdot}106E}{0{\cdot}481}$ is a constant which we will call K.

Then (43) becomes $$K = \frac{L}{y} + \frac{l}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

By simplifying this we obtain $Lx + ly = Kxy$ or $ly = x(Ky - L)$;

whence $$x = \frac{ly}{Ky - L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Substituting this value of x in (43) we obtain

$$W_1 = NLy^2 + nl\left(\frac{ly}{Ky - L}\right)^2 \quad . \quad . \quad . \quad (47)$$

Next differentiate (47) with respect to y. We then have

$$\frac{dW_1}{dy} = 2NLy + 2nl\left(\frac{ly}{Ky - l}\right)\left\{\frac{l(Ky - L) - Kly}{(Ky - L)^2}\right\}$$

$$= 2NLy + 2nl\frac{(ly)(-lL)}{(Ky - L)^3} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

For a maximum or minimum value this is equal to zero. Simplifying (48) we obtain

$$\frac{dW_1}{dy} = 2NLy - \frac{2l^3nyL}{(Ky - L)^3} \text{ and if } \frac{dW_1}{dy} = 0, \quad 2NLy = \frac{2nl^3Ly}{(Ky - L)^3}$$

whence $$N = \frac{nl^3}{(Ky - L)^3} \quad \text{and} \quad \left(\frac{N}{n}\right)^{\frac{1}{3}} = \frac{l}{(Ky - L)}$$

$$\therefore Ky = \frac{l\sqrt[3]{n} + (L \times \sqrt[3]{N})}{\sqrt[3]{N}}$$

and $$y^2 = \left(\frac{l \times \sqrt[3]{\frac{n}{N}} + L}{K}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

We obtain by this equation the minimum value of y.

Since N, n, l, and K are known, the values of the weight per mile of the copper in the junction circuits y^2 is first obtained from (49), and the weight thus found will give the required standard cable equivalent with the minimum weight of copper. The weight of the subscribers' circuit for minimum copper can now be obtained by (46). Note that W_1 is known from (47).

If it is desired to use loaded conductors, we have by (41) $\beta = \frac{0{\cdot}481}{\sqrt{W}}$, and as β is known it is possible to select loaded conductors which give the required β. For any other value of capacity with the same value of β we have

$$\sqrt{\tfrac{1}{2}\omega CR} = \sqrt{\tfrac{1}{2}\omega C_1 R_2} \quad \text{or} \quad \sqrt{\tfrac{1}{2}\omega C \frac{1719}{W}} = \sqrt{\tfrac{1}{2}\omega C_1 \times \frac{1719}{W_2}}$$

or $$\sqrt{\frac{C}{W}} = \sqrt{\frac{C_1}{W_2}} \quad \therefore W_2 = \frac{C_1}{C}W$$

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If W is the weight of copper per mile per circuit required in connection with 0·054, mf., and W_2 the weight required for the new capacity C_1 mf., this becomes

$$W_2 = \frac{C_1}{0{\cdot}054} W \quad . \quad . \quad . \quad . \quad . \quad (50)$$

In connection with the value E, the standard cable equivalent of any cord and wiring losses in the offices concerned in the total connection should not be included. The value E is the estimated standard cable equivalent of the junction line and the equivalent of the subscribers' circuit. The latter, however, includes the standard cord circuit, as explained in Chap. XI.

Minimum Cost Formula for Aerial Lines of Small Gauge.—The problem presented by the cases explained in connection with Fig. 159 occurs also when the lines to be dealt with are aerial circuits. In view of the fact that the formula for the attenuation constant of such circuits is materially different from that for small cable conductors, the development of the formula for minimum cost of aerial lines is materially different from that just dealt with.

In the case of aerial wires of small gauge, if a curve be plotted showing the relationship between weight and standard cable efficiency it will be found to obey the law

$$E = \frac{5}{W^{0{\cdot}82}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

where E is the equated length for a mile loop of conductor the single wire weight of which is W.

Let an efficiency expressed as S standard miles be required between Exchanges 1 and 3 in Fig. 159. Exchange 1 is here supposed to be a zone centre and Exchange 2 a junction centre, then

$$LE + le = S \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

L and l being the respective circuit lengths and e the equated length of 1 mile of the junction circuit between 2 and 3 in Fig. 159. If the price of pole space for a pair is taken to be constant for small gauge circuits—say, P pounds sterling per mile of loop—and x is the price of copper per ton, we have

$$2x\frac{(NLW + nlw)}{2240} + P(NL + nl) = \text{cost} = C \quad . \quad (53)$$

W is the weight per mile of a single wire between 1 and 2, Fig. 159, and w the corresponding weight per mile of a single wire between 2 and 3.

From (51) $$W = \left(\frac{5}{E}\right)^{1{\cdot}22} = \frac{7{\cdot}12}{E^{1{\cdot}22}} \quad . \quad . \quad . \quad . \quad . \quad (54)$$

and $$w = \frac{7{\cdot}12}{e^{1{\cdot}22}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

Substituting these values in (53) we obtain

$$C = \frac{2x}{2240} \times 7{\cdot}12\left(\frac{NL}{E^{1{\cdot}22}} + \frac{nl}{e^{1{\cdot}22}}\right) + P(NL + nl) \quad . \quad (56)$$

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and since $$e = \frac{S - LE}{l} \text{ from (52)} \quad . \quad . \quad . \quad . \quad (57)$$

we have $$C = \frac{2x}{2240} \times 7{\cdot}12\left(\frac{NL}{E^{1{\cdot}22}} + \frac{nl^{2{\cdot}22}}{(S - LE)^{1{\cdot}22}}\right) + P(NL + nl) \quad (58)$$

Differentiating C with respect to E we obtain

$$\frac{dC}{dE} = \frac{2x}{2240} \times 7{\cdot}12(-1{\cdot}22NLE^{-2{\cdot}22} - 1{\cdot}22nl^{2{\cdot}22}(S - LE)^{-2{\cdot}22} \times -L)$$

For a maximum or minimum value this becomes zero,

$$\therefore NLE^{-2{\cdot}22} = nl^{2{\cdot}22}L(S - LE)^{-2{\cdot}22}$$

and $$\frac{N}{E^{2{\cdot}22}} = \frac{nl^{2{\cdot}22}}{(S - LE)^{2{\cdot}22}} \quad \text{whence } \frac{N}{n} = \left(\frac{El}{S - LE}\right)^{2{\cdot}22}$$

and $$\left(\frac{N}{n}\right)^{\frac{1}{2{\cdot}22}} = \frac{El}{S - LE} \quad \text{or} \quad l\left(\frac{n}{N}\right)^{0{\cdot}45} = \frac{S}{E} - L$$

$$\therefore E = \frac{S}{l\left(\frac{n}{N}\right)^{0{\cdot}45} + L} \quad . \quad . \quad . \quad . \quad (59)$$

This equation gives the number of miles of standard cable to be allotted to the circuits between 1 and 2, Fig. 159, and when this value is allotted the minimum cost is obtained. When E is known the weight of copper giving the required efficiency may be found from (54). When E is known e may also be found from (57).

Examples

Example 1.—(Application of formulæ (1) and (19).) An underground cable 100 miles in length is required with a transmission efficiency or effective equated length not exceeding 17 m.s.c., including terminal loss, when a common battery subscribers' line 4 miles in length and of 350 ohms resistance is connected at each end of the circuit. All the circuits are to be of a uniform gauge and the cable is to be fully loaded for side and phantom circuit working. The capacity may be taken as 0·065 mf. per mile loop. What is the cheapest combination of conductor loop and loading coils by means of which the required service may be provided?

First find some known combination of coil and conductor which will meet the requirement irrespective of cost and will give 17 m.s.c. including terminal loss. A mean weight of copper is preferable for this purpose.

As a first step it will be assumed that the terminal loss will be that due to light loading. By Fig. 83, Chap. VII., the terminal loss is 1 m.s.c. with a subscribers' loop of standard cable 4 miles in length. In order to make up for this loss the attenuation length βl to be provided must be equivalent to 17 − 1 = 16 m.s.c., or $\beta l = 16 \times 0{\cdot}106 = 1{\cdot}7$, whence $\beta = 0{\cdot}017$. This can be provided by a 70 lbs. conductor with 0·055 henry per mile. It is known that $\frac{R_1}{L_1} + \frac{G}{C} = 70$ in this

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case, and therefore by formula (48), Chap. VIII., we have

$$\beta = \frac{25{\cdot}3 + (50+20)0{\cdot}55}{2}\sqrt{\frac{0{\cdot}065}{0{\cdot}055}} \times 10^{-3} = 0{\cdot}0158$$

An analysis of previous results shows, however, that the addition of phantom coils of the ordinary type in the side circuit increases β by 5 per cent., and therefore $\beta = 0{\cdot}0158 + 0{\cdot}0008 = 0{\cdot}0166$, which leaves a small and desirable margin between 0·0166 and the limiting value of 0·017 allowed. The *exact* value of R and L for any reasonable value of β could, however, be found without difficulty.

The conductor $R = 25{\cdot}3$ and the inductance $L = 0{\cdot}055$, found in the preceding example, are next adopted as the values of R and L defined in connection with equation (11).

To find the Value of x.—The maximum inductance for R is found from (6); it is

$$L_m = \frac{R}{\frac{R_1}{L_1} + \frac{G}{C}} = \frac{25{\cdot}3}{70} = 0{\cdot}36 \text{ henry}$$

By (7) $$x = \frac{L_m}{L}, \text{ that is, } x = \frac{0{\cdot}36}{0{\cdot}055} = 6{\cdot}5$$

The evaluation of $\frac{BC_1\sqrt{C_1}}{A}$ (see explanation of symbols in formula (1)).—*Computation of* A.—In the case of the 70 lbs. cable under consideration the cost per pair per mile (based on the total cost of the cable divided by the number of pairs in it) is taken at £12. In a phantom combination we have, however, two side circuits and a phantom circuit, and the combination constitutes one unit for cost purposes. We therefore have

$$A = RP_1 = 25{\cdot}3 \times 12 \times 2 = 607{\cdot}2$$

If a cable sheath of fixed internal diameter is assumed which is to be filled as economically as possible, the assumption that the cost per pair will be proportional to the weight of copper will be approximately correct, or if the number of pairs remains fixed and the cable does not vary greatly in diameter the error will not be great. In any case a correction can be made if required, as will be shown.

Computation of B.—Suppose the cost of loading both the side circuits and the associated phantom is £7 per mile, including cost of coils and manholes. (These figures are merely illustrative.)

Now, $B = P_2\sqrt{\frac{x}{R}}$. We have already found that the cost P_2 of loading R is £7, also $R = 25{\cdot}3$ and $x = 6{\cdot}5$. Therefore

$$B = 7\sqrt{\frac{6{\cdot}5}{25{\cdot}3}} = \underline{3{\cdot}55}$$

Computation of C_1.—

$$C_1 = R\left(x + 2 + \frac{1}{x}\right) = 25{\cdot}3\left(6{\cdot}5 + 2 + \frac{1}{6{\cdot}5}\right)$$

$$= 25{\cdot}3 \times 8{\cdot}65, \text{ that is, } C_1 = \underline{219{\cdot}86}$$

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Value of $\frac{BC_1\sqrt{C_1}}{A}$.—From the preceding calculations we have

$$\frac{BC_1\sqrt{C_1}}{A} = \frac{3{\cdot}55 \times 25{\cdot}3 \times 8{\cdot}65 \times \sqrt{219{\cdot}86}}{25{\cdot}3 \times 24}$$

$$= \frac{3{\cdot}55 \times 8{\cdot}65 \times 14{\cdot}83}{24} = 18{\cdot}97 = \left(1 - \frac{1}{x_2^2}\right)(x_2 + 1)^2 \text{ by (19)}$$

The value of x_2 corresponding to 18·97 may be read from Fig. 154; it is $x_2 = 3{\cdot}5$ to the nearest first decimal.

To find R_2 *and* L_2 *from the known value of* x_2.—By (13)

$$R\left(x + 2 + \frac{1}{x}\right) = R_2\left(x_2 + 2 + \frac{1}{x_2}\right)$$

and by (14) this is $C_1 = 219{\cdot}86 = R_2\left(3{\cdot}5 + 2 + \frac{1}{3{\cdot}5}\right) = R_2 \times 5{\cdot}78$

whence $$R_2 = \frac{219{\cdot}86}{5{\cdot}78} = 38 \text{ ohms}$$

This is the resistance of a cable weighing 46·6 lbs. per mile loop.

The inductance L_2 is obtained as follows:—

From (6) $L_{m_2} = \frac{R_2}{\frac{R_1}{L_1} + \frac{G}{C}}$ and therefore $L_{m_2} = \frac{38}{50 + 20} = 0{\cdot}543$

also $\frac{L_{m_2}}{L_2} = x_2$ by (7), that is, $\frac{0{\cdot}543}{L_2} = 3{\cdot}5$

whence $L_2 = \frac{0{\cdot}543}{3{\cdot}5} = 0{\cdot}155$ henry

From the values of $R_2 = 38$ ohms and $L_2 = 0{\cdot}155$ henry per mile, the attenuation constant may be calculated, as the values $C = 0{\cdot}065$ and $\frac{G}{C} = 20$ are known. We have

$$\beta = \frac{\{38 + (50 + 20)0{\cdot}155\}}{2}\sqrt{\frac{0{\cdot}065}{0{\cdot}155}} \times 10^{-3} = 0{\cdot}0157$$

which is very near to the stipulated figure of 0·0158.

The relative cost per mile of the two loading schemes is as follows:—

	£
Cost of Scheme (a)—	
2 pairs in 70 lbs. cable	= 24·0
Loading 2 side circuits and one phantom circuit	= 7·0
	31·0
Cost of Scheme (b)—	
2 pairs weighing 46·6 lbs. per wire per mile $= \frac{46{\cdot}6}{70} \times 24 = \frac{46{\cdot}6}{70} \times P_1 \times 2$ by (12) . .	= 15·98
Cost of loading 2 side circuits and one phantom circuit $= \sqrt{\frac{155}{55}} \times 7$, since the cost varies as the square root of the inductance . . .	= 11·76
(Mem. the loading point is the same as for the side circuit.)	
	27·74

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Difference in cost of the two Schemes—

	£
Scheme (a) per mile	= 31·0
Scheme (b) ,, ...	= 27·74
	3·26

For a 100 pair cable this would amount to £163 per mile or £16,300 per 100 miles.

Now, a 47 lbs. cable is not a standard size, the nearest cable being a 40 lbs. cable. To obtain the required attenuation constant in connection with this conductor an inductance of ·24 henry per mile would be required.

Scheme (c), 40 lbs. conductor—

	£
The cost per mile would be 2 pairs weighing each 40 lbs. per mile per wire $= \frac{40}{70} \times 24 = \frac{96}{7}$	= 13·71
Cost of loading 2 side circuits and one phantom circuit $= 7\sqrt{\frac{0{\cdot}240}{0{\cdot}055}} = 7\sqrt{\frac{48}{11}}$	= 14·63
Total cost . .	= 28·34

The difference between Scheme (a) and Scheme (c) would be—

Scheme (a) per mile	= 31·0
Scheme (c) ,, ...	= 28·34
	2·66

This scheme (c) would, however, require coils of 0·32 henry spaced 1·2 miles apart. This inductance is usually considered too high for practical purposes. If, however, coils of 0·266 henry were spaced 1¼ miles apart the attenuation would be 0·0162, or a little higher than the stipulated case, but it might in the circumstances be accepted.

It is evident by inspection that the use of the 100 lbs. cable would not be justified on the prices, as the cost of the copper would be greater than the combined copper and coils in any of the cases examined.

It is probable that a determination of the cheapest cable on the lines laid down would be sufficiently accurate for a preliminary estimate, but if any more precise figure is required this may be obtained by making a fresh calculation from the data obtained in the preceding case. For example, if it were known that the cost of the 38 ohms cable were £17 per mile instead of £15·98 as assumed, the effect on the cable constants would be as follows.

Example 2.—Find the resistance of a cable pair for minimum cost by formula (19), assuming that the conductor of 38 ohms, already found in example 1, is taken as the known conductor R.

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We have in this case

$$A = RP_1 = 38 \times 17 \qquad = \underline{646}$$

$$B = P_2\sqrt{\frac{x}{R}} = 11{\cdot}76\sqrt{\frac{3{\cdot}5}{38}} = 11{\cdot}76 \times {\cdot}303 \qquad = \underline{3{\cdot}56}$$

$$C_1 = R\left(x + 2 + \frac{1}{x}\right) = 38\left(3{\cdot}5 + 2 + \frac{1}{3{\cdot}5}\right) = 38(5{\cdot}78) = \underline{220}$$

$$\text{whence } \frac{BC_1\sqrt{C_1}}{A} = \frac{3{\cdot}56 \times 38 \times 5{\cdot}78\sqrt{220}}{38 \times 17} = \frac{3{\cdot}56 \times 5{\cdot}78 \times 14{\cdot}83}{17}$$

$$= 18 \quad \therefore \text{ from Fig. 154, } \underline{x_2 = 3{\cdot}43}$$

$$\text{By (14)} \quad R_2 = \frac{C_1}{x_2 + 2 + \frac{1}{x_2}}, \text{ that is } \frac{220}{3{\cdot}43 + 2 + \frac{1}{3{\cdot}43}} = \underline{38{\cdot}4 \text{ ohms}}$$

This value differs so little from the value of 38 ohms previously obtained that there would be no advantage for an approximate estimate in pursuing the matter further, and therefore the values R_2 and L_2 have been found with sufficient accuracy. The method thus illustrated of first finding an approximate result and then taking the results thus found as data for a further problem is calculated to give very exact results. Of course extra trouble is involved, but in view of the magnitude of the monetary interests involved this is a relatively small matter. The whole calculations can be finished in a relatively short time. As a rule, for purposes of approximate estimates the formula is sufficiently exact. The cost of the ducts being constant has been left out of the calculation at this stage, but the effect on the cost per pair can be seen when the required gauge has been found.

Example 3.—Application of the Graphical Method.—The same problem has been here dealt with graphically as in the case of example 1, and the results are shown in Figs. 155 and 156. These curves are arrived at as follows. The values of A, B, and C_1 are calculated as shown in example 1 and inserted in equation (1). The numerical value of the equation is then calculated when x_2 has values from 1 to 10, ten separate calculations being made and tabulated. Fig. 155 shows the variation of price with variation of x_2. It will be seen that the minimum price is obtained when x_2 has the value 3·54. In example 1, x_2 is given as 3·5 to the nearest decimal place. When x_2 is known L_2 may be found from known values of R_2 and a known value $\frac{R_1}{L_1} + \frac{G}{C}$ by equation (8), or

$$L_2 = \frac{R_2}{\left(\frac{R_1}{L_1} + \frac{G}{C}\right)x_2}$$

In this case $\frac{R_1}{L_1} + \frac{G}{C}$ is known to be 70, and therefore $L_2 = \frac{R_2}{70x_2}$. The value of R_2 is obtained from equation (14) from known values of C_1 and x_2 (in this case 3·54 for the critical minimum value). A number of values of R_2 and L_2 are plotted in Fig. 156.

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This graph shows the whole of the possible values of R_2 and L_2 between the limits taken, which will in combination give the required attenuation constant. The values of the resistances are read from the left-hand side of the graph and the inductances on the right-hand side. Any values of R_2 and L_2 which are read vertically will give the required attenuation constant, which is always the same; thus 0·1 henry and 32 ohms gives the same attenuation constant as 0·125 henry and 35 ohms. The advantage of the graphical method is that not only the minimum values are given, but the result of departing from the minimum may be seen by inspection.

Example 4.—An extension of a long-distance circuit is required for a length of 100 miles on existing poles, and it is known that a transmission efficiency equal to that of 400 lbs. aerial conductors is needed for the required service. It is found that a length of 25 miles of 200 lbs. copper aerial circuit is spare and available on the proposed route, and it is required to know whether it would be economical to use this, and also what conductor should be used for the remaining 75 miles to give the required transmission efficiency.

By equation (22) we have

$$Rl = R_1 l_1 + R_2 l_2$$

If R is the resistance of the 400 lbs. conductor per mile loop and l its length, R_1 and l_1 the resistance and length of the 200 lbs. spare wires, R_2 being unknown, we have by substitution in (22) from Table I., Chap. VIII.

$$4{\cdot}4 \times 100 = 8{\cdot}8 \times 25 + R_2 \times 75$$

or $$220 = 75R_2 \quad \text{and} \quad \therefore R_2 = 2{\cdot}93 \text{ ohms}$$

This is the resistance of 600 lbs. copper per mile loop, and it may be verified from the table that 25 miles of 200 lbs. Cu + 75 miles of 600 lbs. copper would give the same attenuation length as 100 miles of 400 lbs. aerial copper. The weights of copper used in the two cases, however, would be as follows.

(*a*) 400 lbs. Cu

$$2 \times 400 \times 100 \text{ lbs.} = 80{,}000 \text{ lbs.}$$

(*b*)
$$\left.\begin{aligned} 2 \times 600 \times 75 \text{ lbs.} &= 90{,}000 \text{ lbs.} \\ 2 \times 200 \times 25 \text{ lbs.} &= 10{,}000 \text{ ,,} \end{aligned}\right\} = 100{,}000 \text{ lbs.}$$

It would evidently be advantageous to erect 400 lbs. copper for the whole distance, even if the 200 lbs. spare wires had to be taken down to provide space on the poles.

Example 5.—A new minor telephone system is to be developed in a given area. The number of the subscribers' circuits is estimated at 500 and their average length at 0·6 mile. The number of junction circuits is estimated at 14 and their length is 14 miles. Assume that an underground system is used throughout.

Ten miles of standard cable is allowed between the subscribers' station and the zone centre when using standard apparatus. The loss due to wiring and coils is estimated at 1 mile of standard cable. The subscribers' apparatus is supposed to be of standard efficiency. The system is local battery. Find the most suitable weight of

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copper for the subscribers' and junction circuits based on the above assumptions.

We have by formula (49) and the fact that $K = \frac{0{\cdot}106E}{0{\cdot}481}$

$$y^2 = \left(\frac{0{\cdot}6 \times \sqrt[3]{\frac{500}{14}} + 14}{\frac{{\cdot}106}{{\cdot}481}E} \right)^2$$

If there is 1 m.s.c. loss in the cords and wiring and the instruments are of standard efficiency the 10 m.s.c. allowed is reduced to 9 and therefore $E = 9$. We then obtain

$$y^2 = \left(\frac{0{\cdot}6 \times 3{\cdot}29 + 14}{0{\cdot}22 \times 9} \right)^2 = \left(\frac{1{\cdot}974 + 14}{1{\cdot}98} \right)^2 = (8{\cdot}07)^2 = 65{\cdot}1$$

this is the weight of copper required for the junction circuits if the capacity is 0·054. If the capacity were 0·065 mf., we have by (50) $\frac{0{\cdot}065}{0{\cdot}054} \times 65 = 1{\cdot}2 \times 65 = 78$ lbs. per mile. Such a circuit efficiency would usually be obtained by loading a lesser gauge of circuit. If the 65 lbs. unloaded circuit were used, or one having the same overall transmission efficiency, the weight of copper for the subscribers' circuits would be obtained from (46), thus

$$x = \frac{ly}{Ky - L}, \text{ that is, } x = \frac{\sqrt{65} \times {\cdot}6}{(1{\cdot}98)\sqrt{65} - 14} = 2{\cdot}467$$

But the weight of copper in the subscribers' line per mile is by definition $x^2 = (2{\cdot}467)^2 = 6{\cdot}07$ lbs. per mile. To prove that the circuits thus provided would give the required efficiency, proceed as follows. By formula (15), Chap. VIII., the resistance of a circuit weighing 65 lbs. per mile per wire = 26·4 ohms per mile loop. As the attenuation assumed is based on the formula $\beta = \sqrt{\frac{1}{2}\omega CR}$, we have $\beta = \sqrt{\frac{1}{2} \times 5000 \times 0{\cdot}054 \times 10^{-6} \times 26{\cdot}4} = 0{\cdot}0597$.

The m.s.c. in the junction circuits is $\frac{0{\cdot}0597 \times 14}{0{\cdot}106} = 7{\cdot}885$ m.s.c. The total amount allowed is (excluding cord and wiring losses) 9 m.s.c. We have, therefore, $9 - 7{\cdot}885$ m.s.c. $= 1{\cdot}115$ m.s.c. available for a length of 0·6 mile of subscribers' circuit = 1·86 m.s.c. per mile. The weight of copper found for these circuits would give an attenuation constant which may be found in exactly the same way as that of the junction circuits; thus $\beta = 10^{-3}\sqrt{289 \times \frac{1}{2} \times 5000 \times {\cdot}054} = 0{\cdot}197 = 1{\cdot}86$ m.s.c., which is the value in m.s.c. left available when the junction circuit efficiency is satisfied. In practice the smallest gauge of wire available is 6½ lbs. per mile, and this would permit of all circuits up to 0·51 mile in length with a wire to wire capacity of 0·08 mf. being provided by that weight of copper. There is, of course, a large element of approximation in using such formulæ, especially as either all underground or all aerial line is assumed; but by using this formula and balancing the advantages obtained by the use of (49), much insight into the necessary transmission requirements is obtained.

It is evident from such studies that subscribers' circuits of very small weight might often be suitable.

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CHAPTER XIII

TRANSMISSION FORMULÆ FOR LINES IN SERIES, WITH APPARATUS IN SERIES AND IN LEAK

Introductory Remarks.—In practical telephony numerous cases involving complex transmission formulæ occur. When, however, the number of finite lines in series exceeds three, the formulæ become so complicated and cumbersome that they are not suited for practical calculation, and they are not here dealt with. See, however, para. 235, *Combined Losses, etc.*

245. To find the Current at the Receiving End of Two Transmission Lines of any Length, in Series.—The received current at the receiving end of a transmission line of any length closed by a resistance Za, is by (17), Chap. III.

$$\frac{E_1}{Z_0 \sinh \theta + Za \cosh \theta} = I_2 \quad . \quad . \quad . \quad . \quad (1)$$

Let the line in Fig. 16 be continued by a second circuit CD, instead of being closed by the impedance Za. If Z'_0 is the characteristic impedance of this line, and θ_1 its line angle, its impedance for any length is, by (6), Chap. III., $Z'_0 \tanh \theta_1$

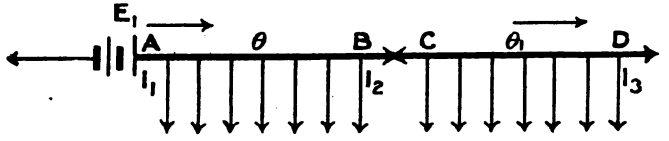

FIG. 160.

If Za in (1) be replaced by $Z'_0 \tanh \theta_1$, we have

$$I_2 = \frac{E_1}{Z_0 \sinh \theta + Z'_0 \tanh \theta_1 \cosh \theta} \quad . \quad . \quad (2)$$

(2) gives the current I_2 at B in Fig. 160, *i.e.* at the point where the line CD of impedance $Z'_0 \tanh \theta_1$ replaces the impedance Za. The current I_2 is also the sending end current in the circuit CD, and by (12), Chap. III.

$$\frac{I_3}{I_2} = \frac{1}{\cosh \theta_1} \quad \text{or} \quad I_3 = \frac{I_2}{\cosh \theta_1},$$

that is $$I_3 = \frac{E_1}{(Z_0 \sinh \theta + Z'_0 \tanh \theta_1 \cosh \theta)} \times \frac{1}{\cosh \theta_1} \text{ from (2)}$$

or $$I_3 = \frac{E_1}{Z_0 \sinh \theta \cosh \theta_1 + Z'_0 \sinh \theta_1 \cosh \theta} \quad . \quad . \quad (3)$$

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246. To find the Current at the Receiving End of Two Transmission Circuits in Series having an Impedance Za at the Sending End.—

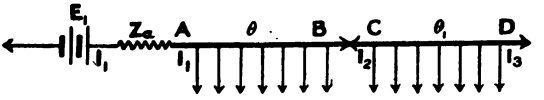

FIG. 161.

If the impedance Zb in Fig. 19 be replaced by a uniform transmission circuit CD, the arrangement becomes as shown in Fig. 161. Replacing Zb in formula (30), Chap. III., by $Z'_0 \tanh \theta_1$, we obtain

$$I_2 = \frac{E_1}{\frac{(Z_0^2 + ZaZ'_0 \tanh \theta_1)}{Z_0} \sinh \theta + (Za + Z'_0 \tanh \theta_1) \cosh \theta} \quad (4)$$

(4) gives the current at the point BC in Fig. 161 as in the previous case, and we again have $\frac{I_3}{I_2} = \frac{1}{\cosh \theta_1}$ or $I_3 = \frac{I_2}{\cosh \theta_1}$ or

$$I_3 = \left(\frac{E_1}{\frac{Z_0^2 \sinh \theta + ZaZ'_0 \tanh \theta_1 \sinh \theta}{Z_0} + Za \cosh \theta + Z'_0 \tanh \theta_1 \cosh \theta} \right) \times \frac{1}{\cosh \theta_1}$$

that is $I_3 =$

$$\frac{E_1}{Z_0 \sinh \theta \cosh \theta_1 + \frac{ZaZ'_0}{Z_0} \sinh \theta_1 \sinh \theta + Za \cosh \theta \cosh \theta_1 + Z'_0 \sinh \theta_1 \cosh \theta} \quad (5)$$

247. To show that the Received Current is the same with the same Voltage if the Sending and Receiving Ends are reversed in Fig. 161.—If the impedance of the uniform circuit CD, Fig. 162,

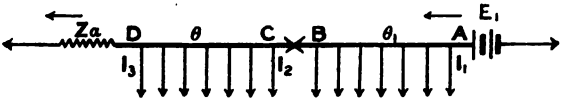

FIG. 162.

and its terminal Za be Zx, we have by formula (17), Chap. III.

$$I_2 = \frac{E_1}{Z'_0 \sinh \theta_1 + Zx \cosh \theta_1}$$

Multiplying by Zx we have

$$ZxI_2 = E_2 = \frac{E_1 Zx}{Z'_0 \sinh \theta_1 + Zx \cosh \theta_1}$$

or

$$E_2 = \frac{E_1}{\frac{Z'_0}{Zx} \sinh \theta_1 + \cosh \theta_1} \quad . \quad . \quad . \quad (6)$$

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but from (17), Chap. III., and Fig. 162

$$I_3 = \frac{E_2}{Z_0 \sinh \theta + Za \cosh \theta} \quad . \quad . \quad . \quad . \quad (7)$$

Substituting the value of E_2 from (6) {Zx is found by (15), Chap. III.}, we obtain

$$I_3 = \frac{E_1}{\frac{Z'_0 \sinh \theta_1}{Z_0\left(\frac{Z_0 \sinh \theta + Za \cosh \theta}{Z_0 \cosh \theta + Za \sinh \theta}\right)} + \cosh \theta_1} \times \frac{1}{Z_0 \sinh \theta + Za \cosh \theta}$$

$$I_3 = \frac{E_1}{Z'_0 \sinh\theta_1 \cosh\theta + \frac{ZaZ'_0}{Z_0} \sinh\theta \sinh\theta_1 + Z_0 \sinh\theta \cosh\theta_1 + Za \cosh\theta \cosh\theta_1} \quad (8)$$

Which is the same as (5).

248. Received Current at the end of Three Transmission Lines of any Length in Series, Closed at the Receiving End.—Let the line angles be θ, $\theta_,$ and $\theta_{,,}$ and the corresponding characteristic impedances Z_0, Z'_0, and Z''_0; these values to be unrestricted.

If in Fig. 162, Za, whose value is unrestricted, be substituted by a uniform line of impedance $Z''_0 \tanh \theta_{,,} = Za$, the current I_3 will be unaltered, and Fig. 163 represents the stipulated conditions.

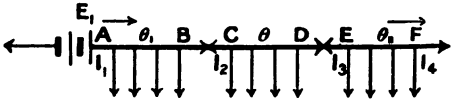

FIG. 163.

By (12), Chap. III., $\frac{I_4}{I_3} = \frac{1}{\cosh \theta_{,,}}$, whence $I_4 = \frac{I_3}{\cosh \theta_{,,}}$. Substitute the value of I_3 from (8) and put $Za = Z''_0 \tanh \theta_{,,}$, we then obtain

$$I_4 = E_1 \Big/ \left(Z'_0 \sinh \theta_, \cosh \theta \cosh \theta_{,,} + \frac{Z'_0 Z''_0}{Z_0} \sinh \theta \sinh \theta_, \sinh \theta_{,,} + Z_0 \sinh \theta \cosh \theta_, \cosh \theta_{,,} + Z''_0 \sinh \theta_{,,} \cosh \theta \cosh \theta_,\right) \quad (9)$$

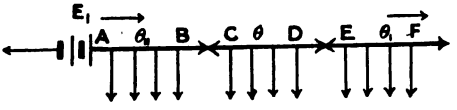

FIG. 164.

It is easily shown that with the same impressed voltage at either end of AF, Fig. 163, the same current is received at the distant end. Fig. 164 shows the alternative case to 163. The line $\theta_{,,}$ is at the sending end in that case, and the line $\theta_,$ at the receiving end. If in (9), $\theta_,$ is now written for $\theta_{,,}$, and $\theta_{,,}$ for $\theta_,$, and if also Z'_0

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replaces Z''_0, and vice versa, it will be found that (9) is unaltered, and the received current is the same. Para. 247 could have been dealt with similarly, but an alternative proof is given.

(9) will next be utilized to find an expression for the received current in a line having a large line angle θ, terminated by two short lines, each having a line angle $\theta_,$. In this case sinh θ and cosh θ in (9) will each be approximately equal to $\frac{e^{\theta}}{2}$, and the expression becomes

$$I_3 = E_1 Z_0 \Big/ \frac{e^{\theta}}{2}\Big\{Z_0 Z'_0 \sinh \theta_, \cosh \theta_, + (Z'_0)^2 \sinh^2 \theta_, + Z_0^2 \cosh^2 \theta' + Z_0 Z'_0 \sinh \theta_, \cosh \theta_,\Big\} \quad (10)$$

$$= \frac{E_1 Z_0}{\frac{e^{\theta}}{2}(Z'_0 \sinh \theta_, + Z_0 \cosh \theta_,)^2} \quad (11)$$

SEPARATION OF THE EFFECTS OF LINE ANGLES AND IMPEDANCES.

Formula (9) shows the combined attenuation due to three lines in series, and their impedances. For comparison with this, suppose that a uniform circuit having the same constants throughout its length, has a line angle $\theta_{,,,}$. Let it be divided into three unequal lengths such that $\theta_{,,,} = \theta + \theta_, + \theta_{,,}$: then by (43), Chap. I., and (10), Chap. III., the received current is

$$I_2 = \frac{E_1}{Z_0 \sinh \theta_{,,,}} = \frac{E_1}{Z_0 \sinh (\theta + \theta_, + \theta_{,,})}$$

$$= E_1/(Z_0 \sinh \theta_, \cosh \theta \cosh \theta_{,,} + Z_0 \sinh \theta \sinh \theta_, \sinh \theta_{,,} + Z_0 \sinh \theta \cosh \theta_, \cosh \theta_{,,} + Z_0 \sinh \theta_{,,} \cosh \theta \cosh \theta_,) \quad (12)$$

This expression only differs from (9) in this chapter in respect of the impedances. The hyperbolic function terms are exactly the same in the two cases, and since (12) clearly includes the development of the sinh of the sum of the angles, the same remark evidently applies to (9). It follows that the attenuation due to these lines in series is the sum of the attenuation in the lines taken separately; and that if two series of such circuits have the same line angles, but different characteristic impedances, the difference in the received currents will be due to the difference in the characteristic impedances.

249. Effect of Inserting a Transmission Line of any Fixed Length, having a Line Angle θ, at any point in a line of any Fixed Length having a Line Angle $\theta_{,,,}$. Both the Line Angles being unrestricted in Value.—The case is shown in Fig. 165.

Let l and l_2 be fixed lengths, and l_1 and l_3 variable lengths, such that the combined length $l - l_2 = l_1 + l_3$. Let the lines AB and EF have the same constants, and the same propagation constant γ. Let the line CD have different constants to AB and EF, such that its propagation constant is γ_1. The effect of moving the line CD to any point on the line ABEF will now be investigated. The

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equation for the received current I_4 is by inspection similar to (9), except that $Z'_0 = Z''_0$ in the case now under consideration. We have therefore

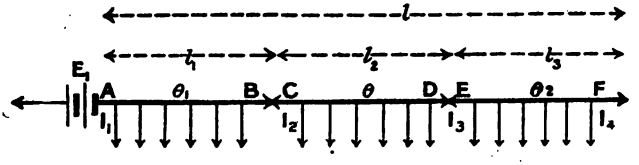

FIG. 165.

$$(a)\quad I_4 = E_1 \Big/ \Big\{ Z'_0 \sinh\theta_1 \cosh\theta \cosh\theta_{11} + \frac{(Z'_0)^2}{Z_0} \sinh\theta \sinh\theta_1 \sinh\theta_{11} + Z_0 \sinh\theta \cosh\theta_1 \cosh\theta_{11} + Z'_0 \sinh\theta_{11} \cosh\theta \cosh\theta_1 \Big\} \quad (13)$$

For purposes of comparison suppose that the section of line CD in Fig. 165 is next fixed at one end of the circuit; then AB and EF will form one continuous length, and the case will be equivalent to that shown in Fig. 160 and (3). If θ_1 in (3) is substituted by $\theta_1 + \theta_{11}$ we shall obtain the equation for the same circuit but with CD, Fig. 160, divided into two parts $\theta_1 + \theta_{11}$; (3) may therefore in that case be written

$$(b)\quad I = \frac{E_1}{Z_0 \sinh\theta \cosh(\theta_1 + \theta_{11}) + Z'_0 \sinh(\theta_1 + \theta_{11}) \cosh\theta} \quad (14)$$

$$= E_1 / \{ Z_0 \sinh\theta (\cosh\theta_1 \cosh\theta_{11} + \sinh\theta_1 \sinh\theta_{11}) + Z'_0 \cosh\theta (\sinh\theta_1 \cosh\theta_{11} + \cosh\theta_1 \sinh\theta_{11}) \} \quad . \quad (15)$$

Comparing (13) with (15), it will be seen that the only difference in the two cases is that $\frac{(Z'_0)^2}{Z_0} \sinh\theta \sinh\theta_1 \sinh\theta_{11}$ in (13) is replaced by $Z_0 \sinh\theta \sinh\theta_1 \sinh\theta_{11}$ in (15).

If $Z_0 \sinh\theta \sinh\theta_1 \sinh\theta_{11}$ be added to, and subtracted from, the denominator of (13), its value will be unaltered, and if the denominator of (15) be called A, then (15) $= \frac{E_1}{A}$ and (13) may be written

$$I_4 = \frac{E_1}{A + \left\{ \frac{(Z'_0)^2 - Z_0^2}{Z_0} \right\} \sinh\theta \sinh\theta_1 \sinh\theta_{11}} \quad . \quad (16)$$

The effect of the position of the circuit CD, Fig. 165, may now be seen. If either $\sinh\theta_1$ or $\sinh\theta_{11} = 0$, (16) becomes $\frac{E_1}{A}$ in each instance. This is the case when CD is at either end of the line, and the received current is evidently the same in both cases. This confirms the conclusion already arrived at. By supposition AF and CD are fixed lengths. It should be noticed that when either θ_1 or $\theta_{11} = 0$, some of the terms of A, *i.e.* of the denominator of (15), disappear; but the value of A nevertheless remains constant,

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because when $\theta_{,}=0$, $\theta_{,,}$ increases to $\theta_{,}+\theta_{,,}$, and when $\theta_{,,}=0$ then $\theta_{,}$ increases to $\theta_{,}+\theta_{,,}$ and this compensates for the terms which disappear.

250. Effect of placing Apparatus Zc at either End of a Uniform Transmission Line terminated by Apparatus at each End. Fig. 19

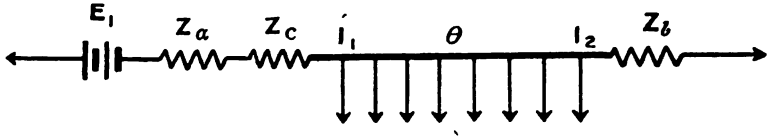

FIG. 166.

and formula (30), Chap. III., show how the received current at I_2 is arrived at in the absence of Zc. If Zc be placed in series with Za, as shown in Fig. 166, then (30), Chap. III., becomes

$$I_2=\frac{E_1}{\frac{Z_0^2+(Za+Zc)Zb}{Z_0}\sinh\theta+\{(Za+Zc)+Zb\}\cosh\theta} \quad (17)$$

If the apparatus is placed in series with Zb at the distant end instead of Za, the formula becomes

$$I_2=\frac{E_1}{\frac{Z_0^2+(Zb+Zc)Za}{Z_0}\sinh\theta+\{Za+(Zb+Zc)\}\cosh\theta} \quad (18)$$

Now (17) and (18) are not the same unless $Za=Zb$, or in other words, unless the circuit is terminated by apparatus of equal impedance Za at both ends before the addition of Zc. If $Za=Zb$ (17) and (18) become

$$I_2=\frac{E_1}{\frac{Z_0^2+(Za+Zc)Za}{Z_0}\sinh\theta+\{Za+(Za+Zc)\}\cosh\theta} \quad (19)$$

251. Effect on the Received Current of an Impedance Z_0 placed at any Point, in Series, in a Transmission Line.—By (17), Chap. III., the received current when the impedance $Z'a$ is at the receiving end of the line is

$$I_2=\frac{E_1}{Z_0\sinh\theta+Z'a\cosh\theta} \quad . \quad . \quad . \quad (20)$$

In Fig. 16, Chap. III., the current at the junction of $Z'a$ and the line is the same as the received current. Now the impedance $Z'a$ is unrestricted in value, and if for $Z'a$ we substitute an equal impedance, viz. $Za+Z_0''\tanh\theta_{,,}$ (where $\theta_{,,}$ is the angle of the added line and $Z_0''\tanh\theta_{,,}$ its impedance), we shall still obtain the current I_2 at the junction of $Z'a$ with the line. The arrangement is shown in Fig. 167. The current at I_2 then becomes by modification of (20)

$$I_2=\frac{E_1}{Z'_0\sinh\theta_{,}+(Za+Z''_0\tanh\theta_{,,})\cosh\theta_{,}} \quad . \quad (21)$$

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The voltage E_1 and the currents I_2 and I_3 indicated in Fig. 167, are related as follows. By (12), Chap. III.

$$\frac{I_3}{I_2} = \frac{1}{\cosh \theta_{,,}} \quad \text{or} \quad I_3 = \frac{I_2}{\cosh \theta_{,,}}$$

that is $$I_3 = \frac{E_1}{\{Z'_0 \sinh \theta_, + (Za + Z''_0 \tanh \theta_{,,}) \cosh \theta_,\} \cosh \theta_{,,}}$$

$$= \frac{E_1}{Z'_0 \sinh \theta_, \cosh \theta_{,,} + Za \cosh \theta_, \cosh \theta_{,,} + Z''_0 \sinh \theta_{,,} \cosh \theta_,} \qquad (22)$$

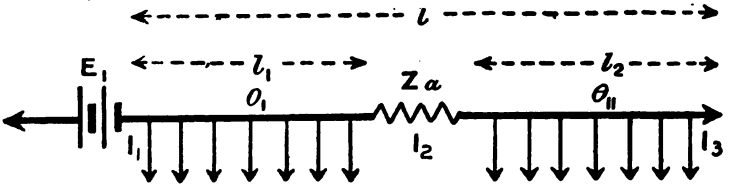

FIG. 167.

This is a general expression for the received current, in the case where an impedance Za is at any point in series between two transmission lines whose line angles are $\theta_,$ and $\theta_{,,}$, and characteristic impedances Z'_0 and Z''_0.

If $Z'_0 = Z''_0$, $\gamma l_1 = \theta_,$ and $\gamma l_2 = \theta_{,,}$, we have the case of an impedance in series at any point of a uniform transmission line of length $(l_1 + l_2) = l$. (22) then becomes

$$I_3 = \frac{E_1}{Z'_0 (\sinh \theta_, \cosh \theta_{,,} + \sinh \theta_{,,} \cosh \theta_,) + Za \cosh \theta_, \cosh \theta_{,,}}$$

$$= \frac{E_1}{Z'_0 \sinh (\theta_, + \theta_{,,}) + Za \cosh \theta_, \cosh \theta_{,,}} \quad . \quad . \quad . \quad . \quad (23)$$

Here $\theta_, + \theta_{,,}$ is the total line angle γl. If this be written θ then (23) becomes

$$I_3 = \frac{E_1}{Z'_0 \sinh \theta + Za \cosh \theta_, \cosh \theta_{,,}} \quad . \quad . \quad (24)$$

This is a general expression for the received current obtained by placing an impedance Za at any point in a uniform line (see Fig. 167).

If in a uniform line $\theta_, = \theta_{,,} = \frac{\theta}{2}$ the impedance is evidently in the middle of the line. In that case (24) becomes

$$I_3 = \frac{E_1}{Z'_0 \sinh \theta + Za \cosh^2 \frac{\theta}{2}} \quad . \quad . \quad . \quad (25)$$

If $\cosh \theta$ in (20) be written $\cosh (\theta_, + \theta_{,,})$, where $\theta_,$ and $\theta_{,,}$ have the meaning already given, then

$$\cosh \theta = \cosh (\theta_, + \theta_{,,}) = \cosh \theta_, \cosh \theta_{,,} + \sinh \theta_, \sinh \theta_{,,}$$

or $$\cosh \theta - \sinh \theta_, \sinh \theta_{,,} = \cosh \theta_, \cosh \theta_{,,}$$

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Substitute this value of $\cosh \theta$, $\cosh \theta_{,,}$ in (24) and we then obtain

$$I_s = \frac{E_1}{Z'_0 \sinh \theta + Za\,(\cosh \theta - \sinh \theta, \sinh \theta_{,,})} \cdot \quad (26)$$

If θ, or $\theta_{,,}$ is zero, (26) reduces to (20).

252. Effect on the Sending End Resistance of an Impedance Za in the Middle of a Uniform Transmission Line.—Fig. 167 illustrates this case, if the lengths l_1 and l_2 are equal and uniform. Let the characteristic impedance of both lengths be Z_0, and the line angle of the total length $l_1 + l_2$ be θ. By (15), Chap. III., if one of the half-line sections be closed at the receiving end by an impedance $\frac{Za}{2}$ its sending end resistance is

$$\frac{E_1}{I_1} = Z_1 = \frac{Z'_0\left(Z'_0 \sinh \frac{\theta}{2} + \frac{Za}{2} \cosh \frac{\theta}{2}\right)}{Z'_0 \cosh \frac{\theta}{2} + \frac{Za}{2} \sinh \frac{\theta}{2}} \cdot \quad (27)$$

If the impedance $\frac{Za}{2}$ is substituted by an impedance

$$\left(Za + Z'_0 \tanh \frac{\theta}{2}\right)$$

in the last equation, we obtain the arrangement shown in Fig. 167, the impedance $\left(Za + Z'_0 \tanh \frac{\theta}{2}\right)$ being that of the impedance Za and the line l_2; we then have

$$\begin{aligned} Z_1 &= \frac{Z'_0\left\{Z'_0 \sinh \frac{\theta}{2} + \left(Za + Z'_0 \tanh \frac{\theta}{2}\right) \cosh \frac{\theta}{2}\right\}}{Z'_0 \cosh \frac{\theta}{2} + \left(Za + Z'_0 \tanh \frac{\theta}{2}\right) \sinh \frac{\theta}{2}} \\ &= \frac{Z'_0\left(2Z'_0 \sinh \frac{\theta}{2} + Za \cosh \frac{\theta}{2}\right)}{Z'_0\left(\cosh \frac{\theta}{2} + \tanh \frac{\theta}{2} \sinh \frac{\theta}{2}\right) + Za \sinh \frac{\theta}{2}} \\ &= \frac{Z'_0\left(2Z'_0 \sinh \frac{\theta}{2} + Za \cosh \frac{\theta}{2}\right)}{\frac{Z'_0\left(\cosh^2 \frac{\theta}{2} + \sinh^2 \frac{\theta}{2}\right)}{\cosh \frac{\theta}{2}} + Za \sinh \frac{\theta}{2}} \\ &= \frac{Z'_0\left\{\left(2Z'_0 \sinh \frac{\theta}{2} \cosh \frac{\theta}{2}\right) + Za \cosh^2 \frac{\theta}{2}\right\}}{Z'_0 \cosh \theta + Za \sinh \frac{\theta}{2} \cosh \frac{\theta}{2}} \\ &= \frac{Z'_0\left(Z'_0 \sinh \theta + Za \cosh^2 \frac{\theta}{2}\right)}{Z'_0 \cosh \theta + \frac{Za}{2} \sinh \theta} \cdot \quad (28) \end{aligned}$$

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If Fig. 167 represents a loading coil section with a loading coil of impedance Za in the centre of the line, the impedance measured from the sending end is given by (28).

If I_1 be derived from (28), and divided by I_3 from (25), we obtain

$$\frac{I_1}{I_3} = \cosh\theta + \frac{Za}{2Z'_0}\sinh\theta$$

This equation is of the same form as (8), Chap. IV.; but it should be noticed that each of the two impedances in this latter case would be $\frac{Za}{2}$ in practice, whilst in the former case the impedance is Za. With this proviso the ratios $\frac{I_1}{I_2}$ and $\frac{I_1}{I_3}$ are the same. It follows from this that the attenuation $\left(\frac{I_2}{I_1}\right)$ in a series loaded section of transmission line earthed at the receiving end, is the same whether the load is in the centre of the section, or split up into two halves placed at its two ends. The received current in the two cases is not, however, the same, because that depends also upon the sending end impedance. Compare (21), Chap. III., with (28) in this chapter.

253. Characteristic Impedance of a Series-Loaded Line.—The required impedance is that obtained at the sending end of a sufficiently large number of sections of series-loaded line, the number being such that the impedance at the sending end is the same whether the receiving end is open or closed. Two cases will be considered: (*a*) when a load is placed at the centre of every section of the line; and (*b*) when half a load is placed at each end of every section.

(*a*) By (7), Chap. III., we have $Z_0 = \sqrt{Z_1 Z}$. Here Z_1 and Z refer to lines of any length and they will be applied to one section of loaded line. In case (*a*) Z_1 is given by (28), and Z, the sending end impedance with the distant end open, is derived from (27) by replacing $\frac{Za}{2}$ by $\left(Za + Z_0\coth\frac{\theta}{2}\right)$. {See (5), Chap. III.} Formula (27) then becomes

$$Z = \frac{Z'_0\left(Z'_0\cosh\theta + \frac{Za}{2}\sinh\theta\right)}{Z'_0\sinh\theta + Za\sinh^2\frac{\theta}{2}}$$

by a similar process to the development of (28). Whence

$$Z_{0,} = \sqrt{\frac{(Z'_0)^2\left(Z'_0\sinh\theta + Za\cosh^2\frac{\theta}{2}\right)\left(Z'_0\cosh\theta + \frac{Za}{2}\sinh\theta\right)}{\left(Z'_0\cosh\theta + \frac{Za}{2}\sinh\theta\right)\left(Z'_0\sinh\theta + Za\sinh^2\frac{\theta}{2}\right)}}$$

By the aid of (52) and (53), Chap. I., this may be written

$$Z_{0,} = Z'_0\sqrt{\frac{2Z'_0\sinh\theta + Za\,(\cosh\theta + 1)}{2Z'_0\sinh\theta + Za\,(\cosh\theta - 1)}} \quad \ldots \quad (29)$$

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This formula is also given by H. W. Malcolm on page 176 of "The Theory of the Submarine Telegraph and Telephone Cable."

(*b*) In this case Z_1 is given by (21), Chap. III., and

$$Z = Za + Z'_0 \coth \theta$$

by inspection of Fig. 18, and (5), Chap. III.

$$\therefore Z_{0,,} = \sqrt{\frac{[\{(Z'_0)^2 + Za^2\} \sinh \theta + 2Za\, Z'_0 \cosh \theta]\,(Za + Z'_0 \coth \theta)}{(Z'_0 \cosh \theta + Za \sinh \theta)}}$$

$$= \sqrt{\frac{[\{(Z'_0)^2 + Za^2\} \sinh \theta + 2Za\, Z'_0 \cosh \theta]\,(Za \sinh \theta + Z'_0 \cosh \theta)}{\sinh \theta\,(Z'_0 \cosh \theta + Za \sinh \theta)}}$$

$$= \sqrt{\{(Z'_0)^2 + Za^2\} + 2ZaZ'_0 \coth \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This is given by G. A. Campbell in the article quoted in para. 76; the symbols are, however, altered in conformity with those here used.

254. Transmission Loss or Gain due to Apparatus in Leak at Either End, or in the Middle of a Long and Uniform Transmission Line.—*Case* (*a*). *Apparatus at the Sending End of the Line.*—Let Za be the impedance of the apparatus, Z_0 the characteristic impedance of the line, and Z the impedance of the apparatus in leak to earth.

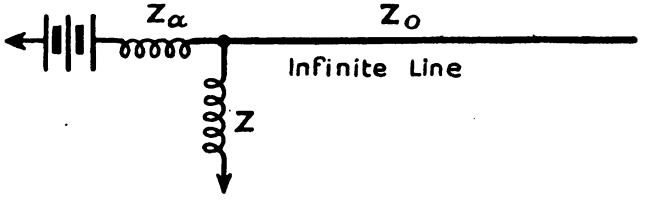

FIG. 168.

The equation for the current I_1 entering the line without the leak is

$$I_1 = \frac{E_1}{Za + Z_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Replacing Z_0 by the joint resistance of Z_0 and Z we obtain

$$I_1 = \frac{E_1}{Za + \frac{Z_0 Z}{Z_0 + Z}} = \frac{E_1(Z_0 + Z)}{ZaZ_0 + ZaZ + Z_0 Z} \quad . \quad . \quad (32)$$

This is the current I_1 entering Za in Fig. 168. Now I_1 divides between Z_0 and Z, and the portion I_2 entering Z_0 is by the law of divided resistances

$$I_2 = \frac{E_1 Z}{Z_0 + Z} \times \frac{(Z_0 + Z)}{ZaZ_0 + ZaZ + Z_0 Z} = \frac{E_1 Z}{ZaZ_0 + ZaZ + Z_0 Z} \quad (33)$$

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The current which entered Za and Z_0 before the addition of the shunt is given by (31). The ratio of (33) to (31) is

$$\frac{I_2}{I_1}=\frac{E_1Z}{ZaZ_0+ZaZ+Z_0Z}\times\frac{Za+Z_0}{E_1}=\frac{Z(Za+Z_0)}{ZaZ_0+ZaZ+Z_0Z} \quad (34)$$

This ratio $\frac{I_2}{I_1}$ may be expressed as an attenuation by the same reasoning as that given in Chap. VII., para. 126. In that case we may write

$$\frac{I_2}{I_1}=e^{-\theta_l} \quad \text{and} \quad \theta_l=\log_e\frac{I_2}{I_1}$$

that is

$$\theta_l=\log_e\frac{Z(Za+Z_0)}{ZaZ_0+ZaZ+Z_0Z} \quad . \quad . \quad . \quad (35)$$

and the loss or gain in miles of standard cable as compared with the circuit without the leak is $\frac{\theta_l}{0{\cdot}106}$.

In view of the fact that the receiving end is too distant to affect the volts or current at the sending end of the circuit, the receiving end impedance does not come directly into the calculation, seeing that both (31) and (32) are equally affected by it. At the receiving end the quantity $\frac{\theta_l}{0{\cdot}106}$ also gives the loss in the received current, due to the leak impedance at the sending end.

Case (b). Transmission Loss or Gain due to Apparatus in Leak at the Middle of a Long Uniform Line.—Fig. 169 illustrates this case.

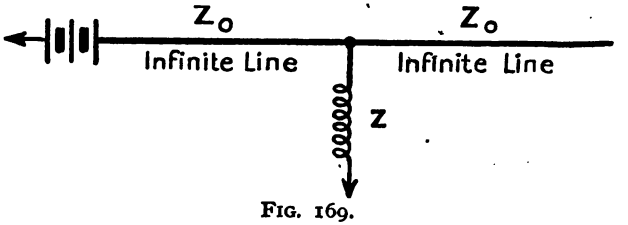

FIG. 169.

This problem is practically the same as the preceding one, inasmuch as by inspection we have a leak between two fixed impedances in each case. The only difference is that in the present case the impedances at each side of the leak are Z_0 instead of one of them being Za. If, therefore, Za be altered to Z_0 in the formula already developed, we obtain from (34)

$$\frac{I_2}{I_1}=\frac{2Z_0Z}{Z_0^2+Z_0Z+Z_0Z}=\frac{2Z}{Z_0+2Z} \quad . \quad . \quad . \quad (36)$$

Case (c). Transmission Loss or Gain due to Apparatus in Bridge at the Receiving End of a Long and Uniform Line.—Fig. 170 applies in this instance. The case is exactly the same as that illustrated by Fig. 168, except that the battery is connected to the opposite end of the impedance system in the two cases. If, therefore, Za be

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changed for Z_0 and Z_0 be changed for Za in (35) the resulting formula will hold good for the receiving end of the line. If the changes indicated are made, however, an inspection of (35) will show that the alterations leave this equation exactly as it was before the changes were made, and it follows that the same effect on the received current is obtained whether the leak is applied to the sending or the receiving end of the line. See Figs. 168 and 170.

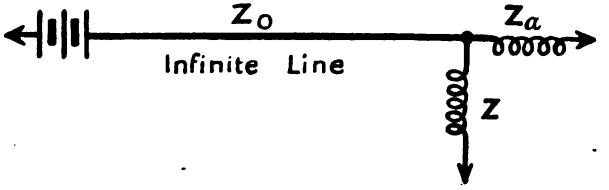

FIG. 170.

Case (d). Relative Loss at the Middle and End of the Line.—The condition which determines whether the loss at the middle or end of the line is the greater is as follows. The ratio of (34) to (36) is

$$\frac{(34)}{(36)} = \frac{(Za+Z_0)Z}{ZaZ_0+ZaZ+Z_0Z} \times \frac{2Z+Z_0}{2Z} \quad . \quad . \quad (37)$$

$$= \left(\frac{2ZZa+2ZZ_0+ZaZ_0+Z_0^2}{2ZaZ_0+2ZaZ+2Z_0Z}\right)$$

$$= \frac{2ZZa+2ZZ_0+ZaZ_0+Z_0^2}{2ZZa+2ZZ_0+ZaZ_0+ZaZ_0} \quad . \quad . \quad . \quad (38)$$

The only difference in the numerator and denominator of (38) is that Z_0^2 in the former is replaced by ZaZ_0 in the latter. If, therefore, the former is greater than the latter the numerator is greater than the denominator, and since in that case (34) is greater than (36) it follows that if Z_0 is greater than Za, the loss at the end of the line due to the leak is greater than that at its middle. On the contrary, if Za is greater than Z_0 the loss at the centre of the circuit is greater than that at the ends.

Experiment shows that in practice a shunt has a somewhat greater effect at the sending end of a line than at the receiving end. In some cases, however, the greater current at the sending end alters the impedance, and this is probably the reason for the greater effect observed at the sending end of the line.

Example.—Calculate the effect on the received current I_4 of placing an impedance $Za = 200 \angle 50°$ in series (*a*) at the end and (*b*) in the middle of a 40 lbs. cable pair having the constants given in Table II., Chap. VIII., and loaded by means of 200 millihenrys per mile. $E_1 = 10$ volts; length of line 40 miles. Z_0 is the characteristic impedance of the line and θ its line angle; θ_1 and θ_2 are the line angles of the sections on either side of the impedance Za. This case is shown by Fig. 167, and the received current may be calculated by (20) and (26).

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The method to be adopted is the same as that utilized in Ex. 4, Chap. IX. We have, therefore,

$$Z_0 = 1756\angle 1^\circ\ 27' \quad \text{and} \quad \theta = (0{\cdot}01626 + j\,0{\cdot}570)40 = 0{\cdot}650 + j\,22{\cdot}8$$

$$\sinh\theta = \sinh(0{\cdot}650 + j\,22{\cdot}8) = 1{\cdot}004\angle 241^\circ\ 27'$$

$$\cosh\theta = \cosh(0{\cdot}650 + j\,22{\cdot}8) = {\cdot}980\angle 211^\circ\ 0'$$

$$\sinh\theta_r = \sinh\theta_s = \sinh(0{\cdot}325 + j\,11{\cdot}4) = {\cdot}974\angle 36^\circ\ 10'$$

$$\sinh^2\theta_r = {\cdot}949\angle 72^\circ\ 20'$$

Case (*a*).—By (20)

$$I_4 = \frac{10}{Z_0\sinh\theta + Za\cosh\theta}$$

$$= \frac{10}{1756\angle 1^\circ\ 27' \times 1{\cdot}004\angle 241^\circ\ 27' + 200\angle 50^\circ\ 0' \times {\cdot}980\angle 211^\circ\ 0'}$$

$$= \frac{10}{1764\angle 240^\circ\ 0' + 196\angle 261^\circ\ 0'}$$

$$= \frac{10}{1948\angle 242^\circ\ 3'} = 5{\cdot}13\angle 242^\circ\ 3' \text{ milliamps.}$$

Case (*b*).—By (26)

$$I_4 = \frac{10}{1764\angle 240^\circ\ 0' + 200\angle 50^\circ\ 0'(0{\cdot}980\angle 211^\circ\ 0' - 0{\cdot}949\angle 72^\circ\ 20')}$$

$$= \frac{10}{1764\angle 240^\circ\ 0' + 200\angle 50^\circ\ 0' \times 1{\cdot}197\angle 109^\circ\ 32'}$$

$$= \frac{10}{1764\angle 240^\circ\ 0' + 239{\cdot}4\angle 159^\circ\ 32'}$$

$$= \frac{10}{1818\angle 232^\circ\ 31'}$$

$$= 5{\cdot}50\angle 232^\circ\ 31' \text{ milliamps.}$$

It will be seen that the difference in the received currents in the two cases is not more than 6½ per cent., but it should be borne in mind that the impedance of the line is very high in comparison with the impedance added to the circuit.

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CHAPTER XIV

THE THERMIONIC VALVE AS A TELEPHONIC RELAY

255. General Considerations.—The term "telephonic relay" is here used to describe the simplest piece of apparatus by means of which the amplification or magnification of the complex electrical waves of speech is produced. This magnification is usually effected at some intermediate point in a telephone circuit, where the volume of speech impressed on the circuit at the sending end is considerably reduced, owing to the impedance and attenuation of the line. As already shown in Chap. V. an electrical complex wave due to speech consists of a fundamental and a number of overtones. Perfect reproduction of speech by a relay involves the reproduction of the fundamental and its overtones in their correct proportion, all the amplitudes being magnified, but their ratio of amplitude being unchanged. It is, therefore, at once evident that the ideal relay must possess great sensitivity, on account of the small forces which are brought into operation, and must be of great rapidity of action, on account of the frequency of the voice currents, which may attain 2000 periods or more per second. It is clear that such an apparatus must practically have no mechanical inertia, and that the amplified currents must be under complete control, beginning and ending instantaneously with the forces which produce them. The thermionic lamp or valve comes much nearer to the fulfilment of all these conditions than any other known device, and this accounts for the fact that it has been generally selected for use wherever telephonic relays have been adopted.

The first attempt to produce a relay, however, was on quite different lines. A telephone receiver was utilized to receive the signals which were required to be amplified; the diaphragm of this receiver was attached mechanically to the diaphragm of a sensitive telephone transmitter; the receiver actuated the transmitter and brought into operation augmented currents, due to the battery in the telephone transmitter circuit. A good magnification was obtained by this means, and considerable work was done in the development of this device, notably by H. E. Shreeve (U.S. Patent 791,656, 1905, and others), and S. G. Brown (British Patent specification 11135, 1908, and many others). The stability and general action of the thermionic valve were, however, ultimately found to be superior to this device, which doubtless accounts for the general adoption of the valve.

The development of the thermionic valve relay is the outcome of modern scientific research, and a brief account of some of the most

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salient facts in connection with its evolution will now be given. It has been known for many years that if the high voltage generated in a Ruhmkorff induction coil be impressed on a partially exhausted vacuum tube a current will flow, although when the tube is filled with air at atmospheric pressure no current passes. In order to pass the current through the partial vacuum, electrodes of platinum were sealed through the glass tube, the electrode connected to the positive terminal of the source of current being called by Faraday the anode, and the negative electrode the cathode. The passing of the current through different gases in these conditions produces beautiful luminous effects, which depend upon the nature of the gas and the amount of the vacuum. In 1878 Sir William Crookes announced his discovery that if the vacuum was greatly increased, the phenomena in the tube were considerably modified, and that in fact violet rays were projected in straight lines from the cathode, and were associated with a dark space around it; he named the rays "cathode rays," and stated that they consisted of very small particles travelling at a high speed, each particle carrying a charge of negative electricity.

It had hitherto been accepted as a scientific fact that the atom was the smallest particle into which any element could be resolved; but Sir J. J. Thomson, in 1897, succeeded in measuring the apparent mass of the cathode particle; he found it to be only one-thousandth part of the lightest atom known, viz. hydrogen, and that it carried a negative charge, $e = 3{\cdot}4 \times 10^{-10}$ electrostatic units. The negative particles were thus brought into greater prominence. They were at first called corpuscles and afterwards electrons, which name they still bear. Subsequent investigations (those of Perrin, 1908–10) give the apparent mass of the electron as 1775 times less than that of an atom of hydrogen, and the charge of the electron in electrostatic units as $4{\cdot}11 \times 10^{-10}$. A continuous supply of electrons may be produced from a heated filament in a partially exhausted, or in a highly exhausted bulb. By this means a permanent electric current can be obtained, if an external path for it is completed, as will now be explained. It will be convenient to assume, in the first instance, that the bulb has been exhausted to a high degree of vacuum, in view of the fact that thermionic relay valves of this type are largely used at the present time for telephonic purposes; and, moreover, the action of such a valve admits of a somewhat simpler explanation than that of a partially exhausted one.

Referring to Fig. 171. In the first instance consider that portion of the diagram shown by the full lines. The filament F in the interior of the bulb (which is usually made of tungsten, owing to the fact that this metal can be heated to a high temperature without deterioration) is connected to a heating battery B_1, and the current circulating through the filament can be regulated by a resistance R. It is found that the filament in these conditions gives off electrons, the amount of which depends upon the temperature of the filament. Professor O. W. Richardson has given the following equation to express the relation between the emission and the temperature of the filament:—

$$N = aT^{\frac{1}{2}}e^{-\frac{b}{T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

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where N = number of electrons emitted per second by a square centimetre of the heated body, T = absolute temperature of the filament, *i.e.* $(273 + t)$, where t is the Centigrade temperature above zero, and a and b are constants depending upon the material of the filament. The term "thermionic" is derived from the heating effects, and is due to Professor Richardson.

In order that a current obeying the law in (1) may be obtained, the anode A and the cathode K, Fig. 171, must be connected by a battery B_2, as shown in the figure, the positive pole being connected to the plate or anode of the valve and the negative pole to the cathode. An electric field is then established between the anode and the cathode, and the negative charges of the electrons are attracted to the anode, whence they circulate in the external circuit and return to the cathode as a unidirectional current, which can be

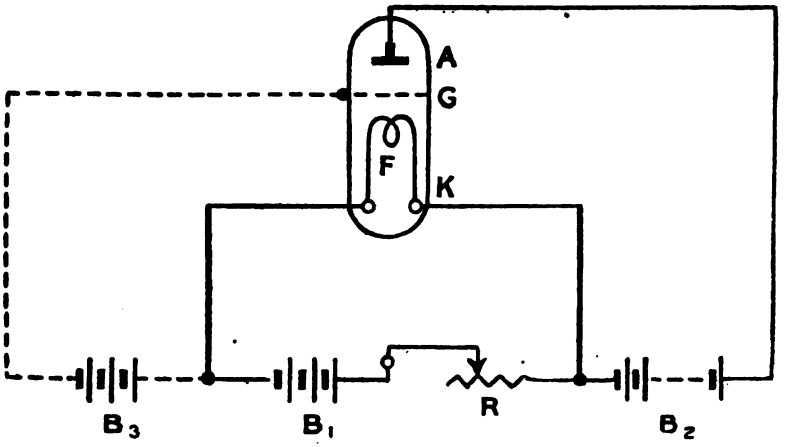

FIG. 171.—Thermionic Valve Circuit for Telephonic Relay Working. Explanatory Diagram.

made evident by a galvanometer in the discharge circuit described. The conventional direction of the electric current is opposite to that of the electron flow. For any given temperature of the cathode there is some minimum voltage in the discharge circuit AB_2K, which is necessary in order to carry off the electrons with sufficient rapidity to maintain a current in accordance with Richardson's law. When this point is reached the thermionic current is said to be saturated. The higher the temperature of the filament the greater is the voltage required for this purpose. Professor Fleming, in 1904, was the first to utilize the fact that a unidirectional current only can be produced by similar means to those described, and he applied the principle to the commercial uses of wireless telegraphy, by converting oscillations to unidirectional currents. A later and important development was discovered by Dr. Lee de Forest, who found that if a third electrode, which was grid shaped, were interposed between the anode and cathode, and if the grid were primed by a suitable voltage derived from a battery as shown at B_3, Fig. 171, then by means

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of the circuit GB_3F a supplementary voltage can be superimposed upon that in the discharge circuit. If a suitable negative value of B_3 is chosen, the current in the discharge circuit is materially diminished; but if an alternating current is then superimposed on the grid circuit—for example, the electric waves due to speech—these waves pass out in the discharge circuit in an amplified form. We have here the essentials of a telephonic relay. It may be noted that the excessively small apparent mass of the electrons offers an inappreciable inertia, and that a suitably designed thermionic valve is sufficiently sensitive to respond instantaneously to the variations of the impressed speech waves within the required limits.

Dr. Irvine Langmuir, of the General Electric Company's Research Laboratory, Schenectady, N.Y., has fully investigated and developed the properties of the type of valve referred to, *i.e.* that in which the pressure in terms of mm. of mercury is reduced to a very low figure (0·000001 mm. of mercury, for example). In this type of valve a pure electron discharge is obtained. (See his paper read before the American Institute of Radio Engineers, April 7th, 1915, an abstract of which appeared in the *Electrician* of May 21st, 1915.) It may here be mentioned that in vacuum valves having a much greater mercury pressure, the electrons in their passage through the lamp ionize the molecules of gas, *i.e.* the molecules are split up into ions, and electrons are released which react on the electron discharge and complicate the phenomena. These valves are known as "soft" valves; they have in the past been largely used in pioneer relay work, but at present the "hard" type is preferred on account of its greater stability. These latter valves are not subject to the ionizing effect referred to. In order, however, to exhaust thermionic valves suitably Dr. Langmuir found it necessary to take special steps to ensure that any gas occluded in the glass and metal filament, etc., are removed. It may be mentioned that the anode and grid are frequently made of nickel. Dr. Langmuir has given the name of pliotron to the type of valve with three electrodes and an extremely high degree of vacuum as developed by him. Experiment shows that a definite relation exists between the heating current, the grid voltage, and the discharge voltage, in order to secure the best results. The law of emission of electrons with variation of temperature has already been given. Dr. Langmuir's researches show that with a high exhaustion of the valve, the maximum current which can be produced in the discharge circuit in the absence of the grid is proportional to the potential between the anode and cathode raised to the 1·5th power. When the grid is added the relation found by Dr. Langmuir is expressed by the following equation:—

$$i = A(Va + KVg)^{1.5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where i = the current flowing to the anode
Va = the anode voltage
Vg = the grid voltage

A and K are constants which depend upon the dimensions of the valve and the form and position of the electrodes. It should be noticed that if the potential of the grid is negative with respect to the filament a current cannot flow in the grid circuit, since by supposition the potential of the grid must be positive with reference to the cathode in

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order to attract the negative charge of the electrons. Voltage changes impressed on the grid can therefore be effected with a negligible current in that circuit, and since the power absorbed is proportional to (3), a very small expenditure of power due to speech waves in this part of the circuit can in accordance with (2) be made to produce a considerable change in the voltage and current of the discharge circuit.

256. Determination of Telephonic Relay Characteristics.—The valves described have been developed for the purposes of wireless telegraphy, and for these purposes they generally require a small input of power and a high magnification. The requirements of thermionic valves for telephonic purposes are somewhat different, the required power being more, and the magnification less, than in the wireless case. The power required determines the size of the valve; the best voltages for the discharge circuit, for the heating or filament circuit, and the grid circuit, are determined by a series of actual tests. Details of the methods of test will be found in a paper by Messrs. C. Robinson, B.A., and R. M. Chamney, A.M.I.C.E., of the Engineer in Chief's Department, Post Office, entitled "Telephonic Relays and their Application to Commercial Circuits," read before the Post Office Institute of Electrical Engineers and included in the Professional Papers of that body. The tests were devised by the authors, and the following graphs of results are taken from the paper. Fig. 172 shows an important characteristic, *i.e.* the variation of the current flowing between the anode and cathode, and in the grid circuit, due to variations of the voltage in the battery B2 for different values of B3 (Fig. 171). The slope of these curves shows how the current varies with the voltage, and the corresponding resistance at any point of the curve can be deduced from it. The straight part of the curves indicates a resistance of 26,300 ohms, the lamp under test was a "hard" one. Now in actual practice the amplified current in the discharge circuit is transmitted by a transformer to the outgoing telephone circuit to which the magnified current is supplied, and the best impedance of the winding of the transformer connected in the discharge circuit is the same as that of the lamp discharge circuit itself for maximum absorption of power in the transformer. Similarly the winding of the transformer connected to the outgoing telephone circuit should be the same as that of the line (see Chap. VII). It may also be seen from the curve in question that an increase in the discharge volts up to about 250 volts produces an increase in the discharge current. At this point it is probable that the electron emission from the cathode is completely disposed of by the voltage, and that there is therefore no advantage in increasing it further,—this is shown by the flattening out of the curve for the higher values of V. It is not necessary in all cases, however, to use the highest possible voltage, as in some cases the required magnification can be obtained with a less voltage. The grid current is also shown in the figure. It is in all cases very small and is associated with a + grid voltage. Fig. 173 shows the variation of the discharge current and the grid current, *i.e.* the current circulating in the circuit shown by the dotted lines in Fig. 1, with variations of grid potential. It is seen from this curve, as already mentioned, that when the voltage

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of the grid is negative with reference to the filament, no current flows in the grid circuit; this is clear from the fact that the grid current entirely disappears in the curve for all values between

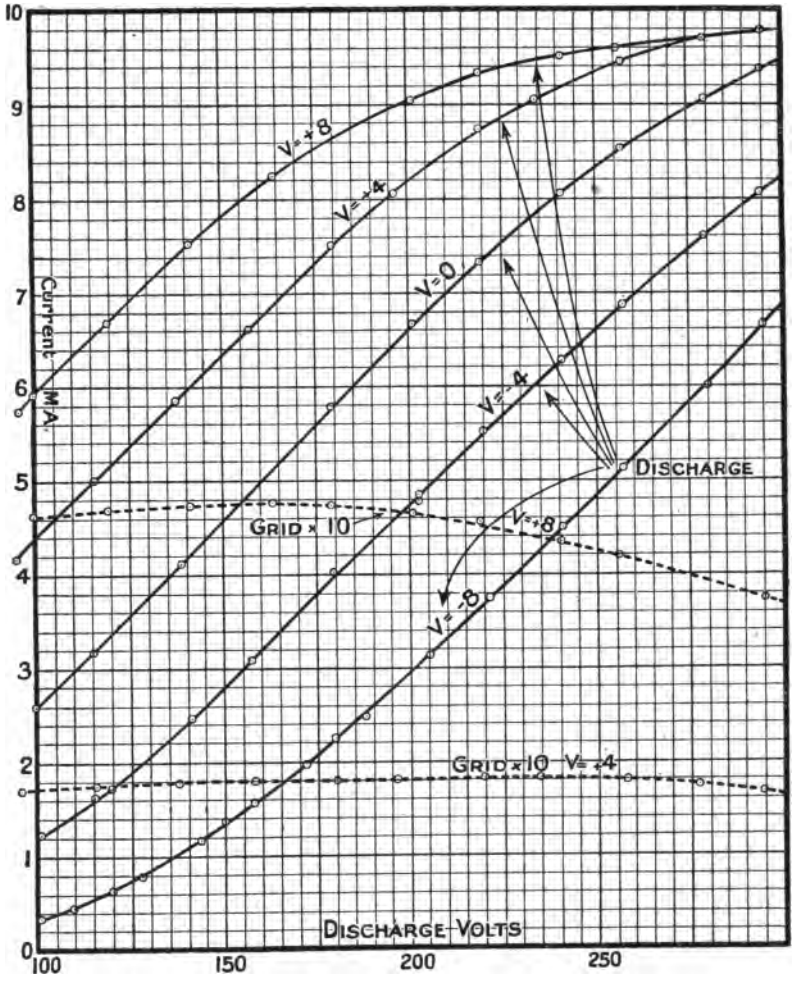

FIG. 172.—Variation of Current flowing between Anode and Cathode, and in Grid Circuit, due to Variations of Discharge Volts and Grid Volts.

potential 0 and −15 in the grid circuit. The slope of the curve showing the discharge current with change of grid voltage should be straight and steep, to give the best result, because such a curve shows by its steepness that a small change in the grid voltage produces the maximum current change in the discharge circuit, and

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the fact that the line is straight indicates that the increase is equal whether the voltage is increased or decreased by an equal amount. If, therefore, a small alternating voltage, which of course alternately

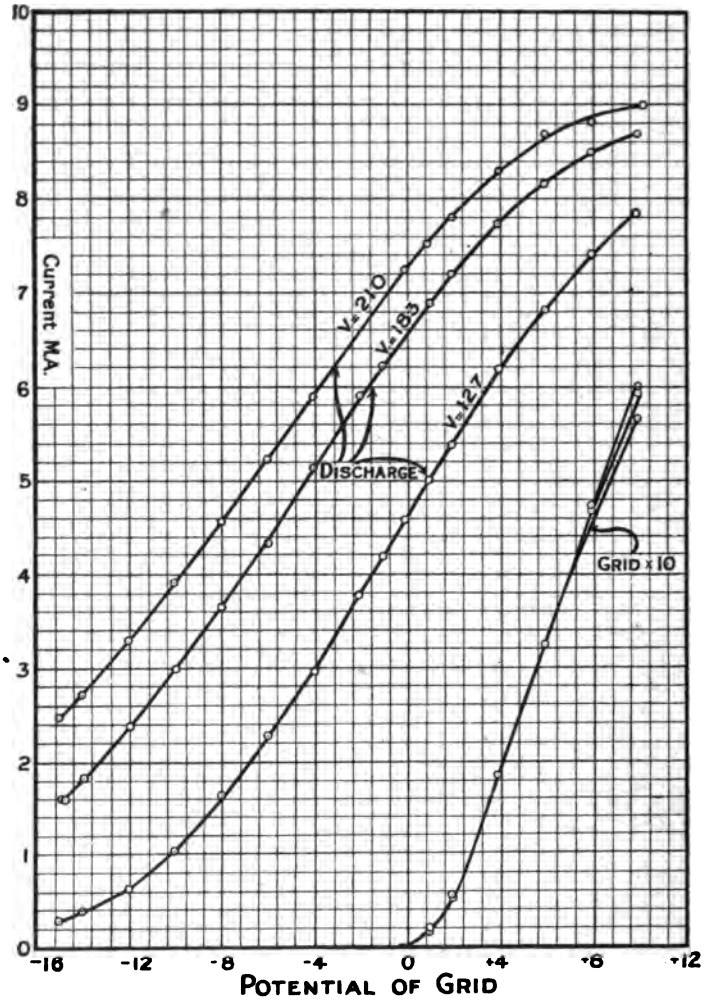

FIG. 173.—Variation of Discharge Current and Grid Current with Variation of Grid Potential.

increases and decreases from a neutral point, were impressed on the grid, it would not suffer distortion. Now such an alternating voltage is impressed on the grid by the speech waves which enter the relay on that side. If, as an example, 210 volts were used in the discharge circuit, then the straight portion of the 210 volt curve in

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Fig. 173 would be chosen, and this would be obtained by a priming negative grid voltage between 0 and −15; this need not be more

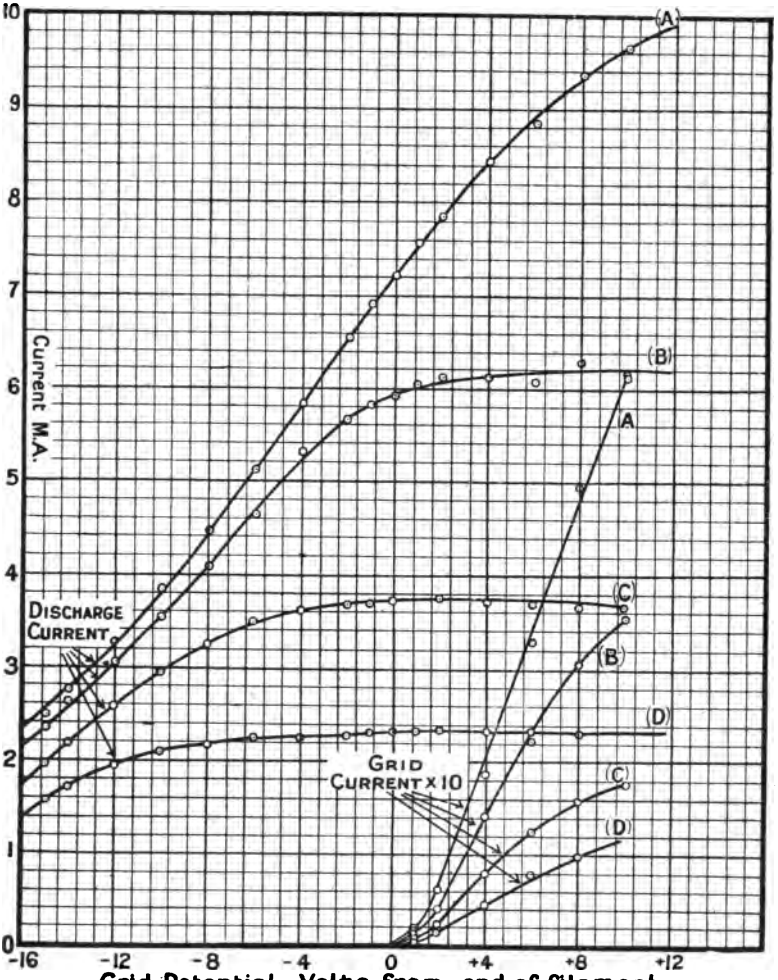

FIG. 174.—Effect on Discharge Current and on Grid Current of varying the Heating Current I in the Cathode Filament.

Curves marked (A) I = 0·73 amp.
,, ,, (B) I = 0·70 ,,
,, ,, (C) I = 0·675 ,,
,, ,, (D) I = 0·65 ,,

than 3 or 4 volts, as the same result is obtained with these values as for larger negative voltages. It is further obvious from the

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fact that no current is circulating in the grid circuit that the apparent impedance of that circuit is very high; when a transformer is added on the input side to connect the relay to the telephone line, as is always the case, the condition of no current is slightly modified, but the impedance is very high and the winding of the transformer connected to the input side should be correspondingly high on the principle already laid down. Fig. 174 shows the effect on the discharge current and on the grid current of varying the heating current in the cathode filament. The slope of the current in the discharge circuit should be steep and it should be straight, for the reasons already explained. This is attained with a heating current of about 0·72 ampère.

Similar curves to those given are required for every different type of lamp, and for every different maker's products.

It will be seen that information is obtained by these curves as to the best voltage for the batteries B_1, B_2, and B_3. We also learn that the input transformer should be a step-up transformer and the

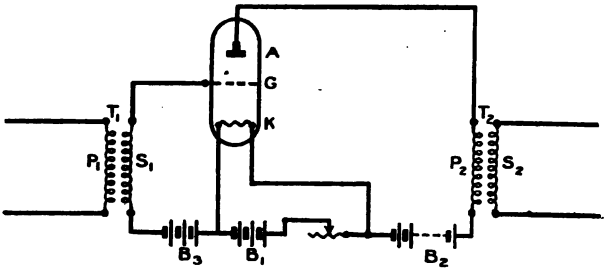

FIG. 176.—Simple Telephonic Relay Circuit, arranged for speaking from T_1 to T_2.

output a step-down transformer, usual values in the former case are a ratio of 30 to 1 and in the latter case 10 to 1.

From the high impedance in the grid circuit it will be seen that small cells of high resistance would be suitable. This would also appear to be the case as regards the discharge circuit, but there is some risk of overhearing if high resistance cells are used; and, moreover, secondary cells are often found to be convenient. From the magnitude of the current circulating in the filament circuit secondary cells are necessary.

A photograph of a "hard" lamp, which is often called a lamp of the "French" type, as used by the Post Office, is given in Fig. 175A and 175B. The cathode, grid, and anode are indicated by the letters K, G, A.

Measurement of Magnification.—This measurement is made by connecting an alternating current to the primary side of the transformer T_1 which is connected to the grid circuit (see Fig. 176), and measuring the watts entering the relay in that primary, and also the watts leaving the relay circuit on the secondary side of the transformer T_2. The Franke machine is frequently used for this purpose (see Chap. X.). In measuring the watts on the input side the voltage drop V_1 is first measured at the ends of a non-inductive

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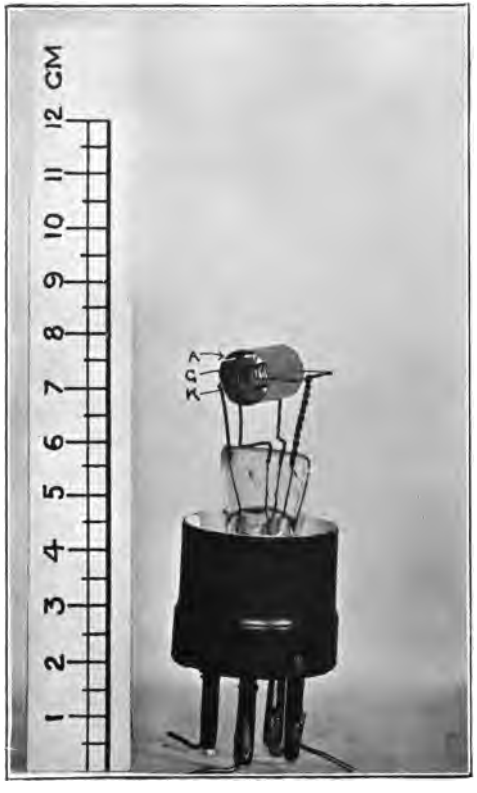

FIG. 175A.

FIG. 175B.

Thermionic Valve used by the Post Office. Two views. A = anode, G = grid, K = cathode.

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resistance R_1 in series with the primary winding, and then the voltage drop V_2 at the ends of the primary winding is taken. The current in the primary winding is then evidently $\frac{V_1}{R_1}$ and the volts are measured by V_2. The voltage is out of phase with the current by an angle ϕ which is measured. Then the power P is

$$P_1 = \frac{V_1}{R_1} V_2 \cos \phi \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the output circuit, if the power is absorbed by a non-reactive resistance R_2, and this is the only resistance in series with the secondary winding of T_2, it suffices to measure the voltage drop V_3 at the ends of the resistance, which also corresponds with the ends of the winding, we then have

$$P_2 = \frac{V_3^2}{R_2} = V_3 I \text{ watts} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $I = \frac{V_3}{R_2}$.

The magnification of power is then obtained by dividing (4) by (3). It should be noticed that inasmuch as there is usually no current in the grid circuit, the foregoing is as much a test of the input transformer as of the valve.

Fig. 177 shows the results of a series of tests taken in this way on a "hard" valve. If it be assumed that the speech current entering a telephone circuit at the sending end has a mean value of 0·005 ampère, and that the circuit has a characteristic impedance of 1000 ohms, then the input watts in this case would apply to the case of a telephone circuit of the longest conceivable length as one limiting value and one having an attenuation length equivalent to 30·3 miles of the standard cable as a lower limit, these limits corresponding to the input values 0 and 40 respectively, as given in Fig. 177. For a circuit of lower impedance the equivalent attenuation length would be somewhat less. It will therefore be seen that this valve gives its maximum magnification when fixed at points in a long line which have an attenuation greater than 30·3 miles of standard cable in the conditions supposed. It should be borne in mind, however, that the current at the beginning of the trunk line may be considerably less than the 5 milliampères stipulated, either on account of attenuation in the subscribers' lines, or due to a telephone below the normal efficiency, or a subscriber speaking in an abnormally low voice, etc., and all these factors must be taken into account in determining the most suitable input to be provided for by the valve. It will be seen from Fig. 177 that the maximum magnification is given when the input is very small, *i.e.* about five microwatts, and that as the input increases from that figure to 40 microwatts the magnification decreases. By this means a variation of volume of speech within these limits is to some extent corrected. This variable magnification does not mean that harmonics of small amplitude will be magnified in a different ratio to the much larger volume of the fundamental, for both harmonic and fundamental simultaneously form part of the total value of the input energy, and it is this total value

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which determines the magnification; so that all the instantaneous frequencies in the input energy are magnified in the same proportion, so far as this curve shows. The relative magnification of high and low frequencies can, however, be modified by the design of the input transformer. In order to appreciate the significance of the values of

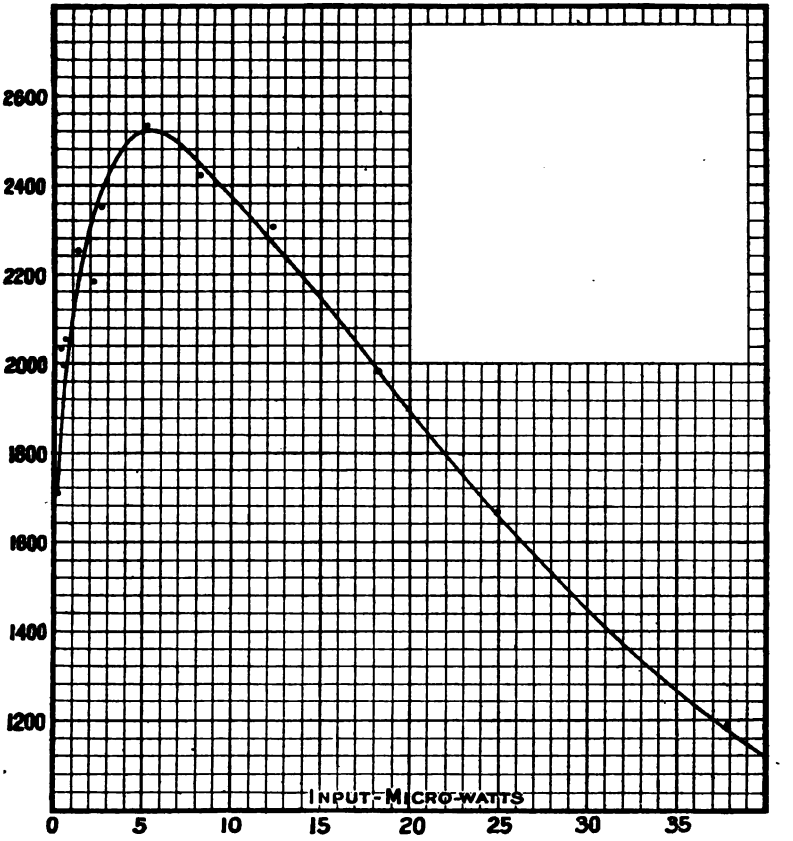

FIG. 177.—Variation of Magnification with Input.

Heating Current	.	0·73 amp.
Discharge Volts	.	230
Grid Volts	. .	−5·7 from end of Filament.

magnification given in the curve, say for example a magnification of 1000, we have only to take a representative telephone circuit and find some point distant from the sending end where the power has diminished by attenuation to one-thousandth part of the impressed power. This attenuation is such that the magnification of 1000 would entirely annul it and restore the power to its sending end

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value ; the magnification may therefore be said to give an improvement equivalent to that attenuation length, and it may be conveniently expressed in terms of standard cable. If as an example we again assume the current at the sending end to be 0·005 ampère and the characteristic impedance of a very long loaded line 1000 ohms without phase angle between volts and current, we have at the sending end

$$P = VI = Z_0 I^2 = 1000 \times (0{\cdot}005)^2 \quad . \quad . \quad . \quad (5)$$

At some distant point on the line we have by supposition

$$P_1 = Z_0 I^2 e^{-2\beta l} = 1000 \times (0{\cdot}005)^2 e^{-2\beta l} . \quad . \quad . \quad (6)$$

Here $2\beta l$ has a value such that the ratio of (5) to (6) is $\frac{P}{P_1} = \frac{1}{e^{-2\beta l}}$ $= 1000$, or $e^{-\beta l} = \sqrt{0{\cdot}001} = 0{\cdot}0316$, whence $\beta l = 3{\cdot}45$ and $\frac{3{\cdot}45}{0{\cdot}106}$ $= 32{\cdot}5$ miles of standard cable. Similarly a magnification of 2000 would correspond to an improvement equivalent to nearly 36 miles of the standard cable. Comparing these figures with the input values previously analysed, it would appear that the improvement obtained would be a little more than the actual power attenuation due to the lines connected to the input side of T.

It must be borne in mind, however, that the improvement values are only rigid for one frequency, and over a very small range of input values. This is in fact the principal limitation of a lamp of this capacity, but it is found to meet commercial requirements. Further investigation is, however, being made by the Post Office Research Section on this point.

257. Telephonic Relay Systems.—(*a*) *Single Relay Repeaters.*—In order to introduce a thermionic lamp relay into a telephone circuit in the simplest way, *i.e.* to provide for speaking in one direction only, transformers are added, as already mentioned, to the relay, see Fig. 176. The transformers perform the function of confining the battery currents in a local circuit, and if they are suitably designed they permit of the power generated being utilized to the greatest advantage. It may be noticed, however, from Fig. 176 that a voltage can only be impressed on the grid from the side of the line to which it (the grid) is adjacent, a current entering the relay by the transformer T_2 is not effective for operating the grid. It is however, of course, an indispensable condition for commercial telephony that speech shall be possible in both directions, and to make this possible a device originated by Edison is used. In order to facilitate the explanation of this device Fig. 178 should be examined. This arrangement fulfils exactly the same object as Fig. 176, and the similarity is evident. In this case, however, the primary is divided into four equal windings as shown, and a resistance is inserted in the middle of the primary coils ; this reduces the current in the transformer, but does not in any way affect the principle of the arrangement. If now the line which was previously connected to the secondary of T_2 be substituted for the resistance R, and if its impedance is the same, the electrical conditions will not be in any way affected ; speech can now be conducted between the two ends of the line and the speech currents are amplified in the relay circuit,

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but this does not benefit the line because the discharge side of the relay is not yet connected to it. If, however, the secondary side of the transformer T_2 be joined to the line through the primary windings

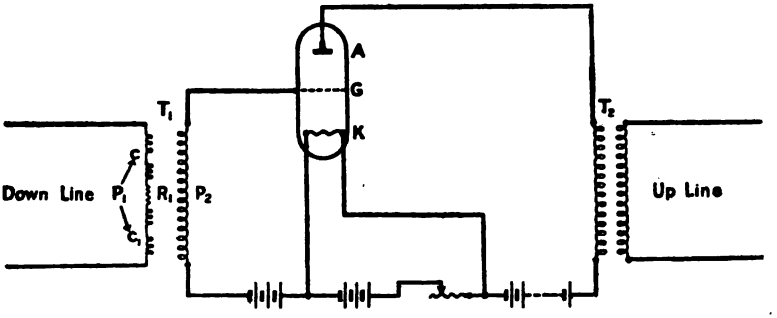

FIG. 178.—Single Relay Repeater. Explanatory Diagram.

of T_1, as shown in Fig. 179, the magnified currents are repeated into the line, they split through the transformer windings in parallel, and the repeated currents are propagated to *both* ends of the line. If the resistances on each side of the point of connection of the relay are equal, the secondary winding of T_1 will not be affected, as there will

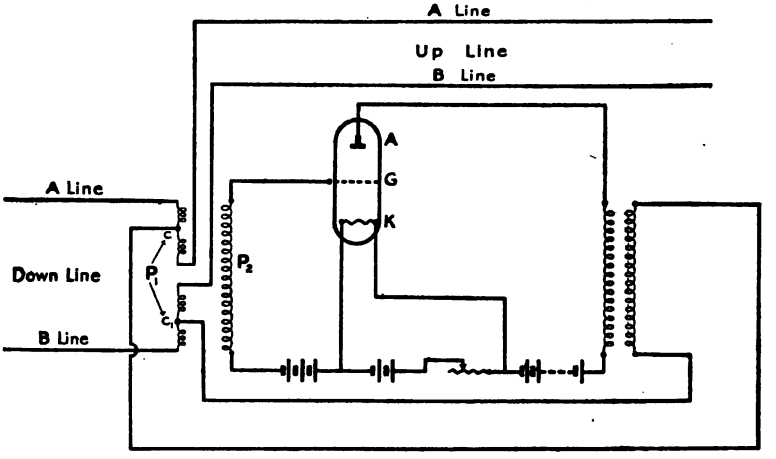

FIG. 179.—Single Relay Repeater. Explanatory Diagram.

be no magnetic effect in the primary owing to the relay currents circulating through the windings in opposite directions; such an arrangement as that shown in Fig. 179 is frequently called a telephonic repeater to distinguish it from a telephonic relay as previously defined. The essential feature of this device (Fig. 179) is that the relayed currents must be repeated into the line without affecting the grid

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circuit, and this can only be done if the line impedances on both sides of C and C_1, Fig. 179, are equal. If such is not the case there will be a resultant magnetic effect on the secondary windings of T_1, and the resulting current will actuate the grid, this will give rise to a magnified current in the discharge circuit, which will again circulate in the primary circuit of T_1, and the out-of-balance current will once more give rise to a current in the secondary of T_1 and the grid. This cycle of events will continue, and in this way the relay will be continuously actuated. The rapidity of the impulses actuating the relay will depend on the electric constants of the circuit, which will oscillate at its characteristic frequency. If the effects are sufficiently powerful a loud note will be heard; this effect is usually known as "*howling*," and it prevents the proper use of the relay.

Problems of Balance in the Single Relay Repeater.—We are here face to face with one of the most difficult problems of telephonic relay working, *i.e.* that of maintaining an equal impedance in the up and down lines attached to the relay, for all the important frequencies which enter into the composition of speech. It is not sufficient to prevent the relay "howling;" if there is a want of balance which is not sufficient to produce that result, the parasitic speech waves which enter the relay will be added to those which are being amplified in the usual way, and may produce serious distortion and bad articulation. Want of balance in the line circuits may be produced by a number of causes, one of the most obvious of which is the change of impedance with length. If the line is not sufficiently long, a change in the circuit conditions at the distant end—such, for example, as opening, or closing, or extending the circuit—will alter its impedance, and at once throw the valve out of balance. If, however, the line is sufficiently long from the electrical point of view, then no matter what change is made at the distant end, the apparent impedance of the line as measured from the relay end is not affected. A reference to Fig. 73 shows the effect on the impedance of varying the length of a 70 lbs. cable circuit. It will be seen from that figure that the sending end impedance becomes practically constant with a length of 30 miles of standard cable. It is convenient to regard the amplified currents which affect the relay, due to varying impedance, as being sent out from the relay, and reflected back to it from the distant end, the current being thus propagated over twice the length of the line. Now experience shows that if the electrical length of the line is equivalent to 15 miles of standard cable, a relay lamp of normal sensitiveness is not affected, and taking the double journey due to reflection into account, this agrees with Fig. 73. The magnitude of the characteristic impedance of the line modifies the result to a small extent, but the rule given is found to be of practical utility. Other causes of want of balance are, different composition of the lines on the two sides of the relay, or apparatus bridged across one side (up or down) and not the other, or imperfect insulation, etc. If the balance of the lines cannot be restored, it is necessary to reduce the sensitiveness of the repeater system by shunting the relay; this can be done by means of a high resistance shunt across the grid-anode circuit. The resistance can usually be varied by steps of 50,000 ohms, to 600,000 ohms; a very small capacity is, however, sometimes used as a shunt. The lines

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may also be to some extent balanced by means of a sliding resistance inserted on either side of the relay. In open-line working it is sometimes necessary to reduce the sensitiveness of the relay to prevent its being affected by currents induced in the line by power circuits and other sources of disturbance. The net result of the various causes of unbalance referred to is that a magnification equivalent to the elimination of 30 miles of standard cable as deduced from Fig. 177 can very rarely be maintained. The ideal position for a single relay is in the middle of electrically long underground conductors having a total electrical length not less than that equivalent to 30 miles of standard cable. To provide an entirely stable service, however, the relay should not be worked so that it is operated by excessively small inputs; in actual practice in double-repeater systems an improvement equivalent to from 15 to 20 miles of standard cable is usually arranged for, but on relatively short lines it may not be possible to attain this figure. In some exceptional cases a single relay in a very long line may have an improvement equal to about 27 miles of standard cable.

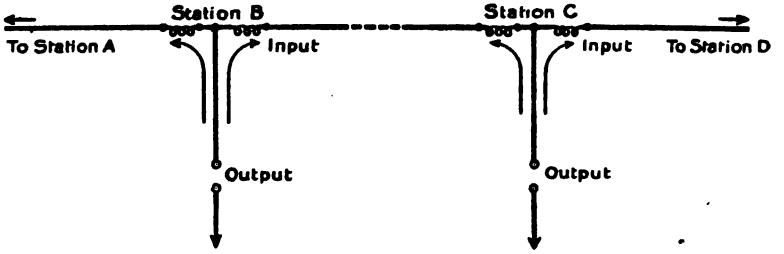

FIG. 180.—Single Relay Repeater Working. Explanatory Diagram of Circuit Arrangement.

Interaction between Relays in the Same Circuit.—If the circuit is of such an electrical length that two or more relays are necessary, a further difficulty arises, for if the valves are not sufficiently far apart they interact on each other, unless special arrangements, to be shortly described, are taken to prevent this. The difficulty is caused by the fact that the amplified energy in each relay system travels both backwards and forwards along the line as shown in Fig 180, where the arrows indicate the direction of flow of the amplified currents. If the distance between the repeaters is not sufficient, the amplified speech waves, from B for example, on arrival at C will be reflected back to B, and will affect that valve a second time. If the effect is sufficiently powerful serious disturbance to working may be caused by this means. Experience shows that the magnification of any valve should not be greater than that necessary to permit of the transmitted power arriving at any relay, being of the same value in all cases, assuming a uniform line and equal spacing. This means that the attenuation of any section of line must in all cases annul the effect of the magnification in the preceding valve, and this rule applies whether the spacing is equal or not. It should be noticed that if this theoretical scheme can be perfectly carried out transmission

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to any required distance can be maintained, because the effects of attenuation are continually neutralized. In order for this to be possible, each valve must also correctly reproduce the impressed speech waves in relative amplitude without distortion. If the line is uniform throughout and divided into equal relayed lengths, the total attenuation of any number of lengths with the power regulated as stated, would be that of one length only. Generally speaking, this is theoretically possible in double-repeater systems, but it could not be attained at present in single-relay repeater systems.

(*b*) *Double Relay Repeaters.*—On account of the difficulties in balancing the impedances on each side of a single telephonic repeater, especially where more than one such repeater is used in a telephone circuit, the line is divided up into isolated lengths, two repeaters being allotted to each junction of separate lengths, as shown in Fig. 181; this arrangement replaces that shown at B or C in Fig. 180. In order to neutralize the amplified current which splits through the

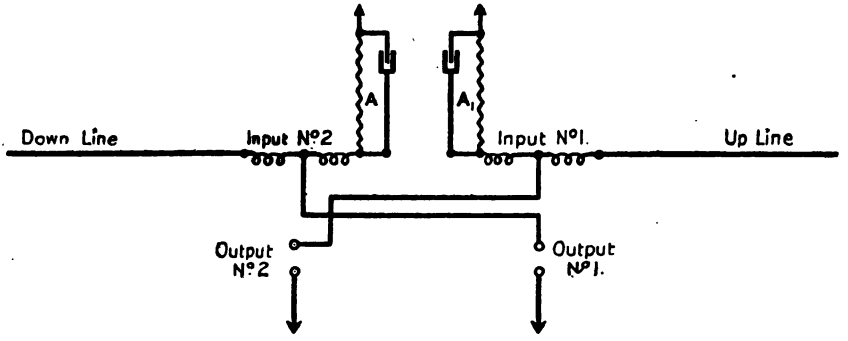

FIG. 181.—Double Relay Repeater Working. Explanatory Diagram of Circuit Arrangement.

primary on the input side of the transformer (P_1, Fig. 179), artificial balancing circuits are utilized as shown at A and A_1, Fig. 181; the secondary side of the output circuit on the one side is joined as shown to the centre of the input transformer on the other side, thus the first relay repeats into the second and passes on the amplified current to the next section of line; the arrangement is shown in greater detail in Fig. 182. The switches S_1 and S_2 are provided for reversing the direction of the current; this is found to be of advantage because of the fact that the current in one part of the apparatus tends to get out of phase with that in other parts of the system, and the reversal of the current is frequently beneficial from the phase point of view. *The advantages of the double relay repeater are:* (*a*) Each section of line can be balanced independently and does not depend on the similarity of the adjoining section. (*b*) The artificial balancing arm of the combination is absolutely stable, and therefore tends to stabilize the arrangement, one possible variable being replaced by it in each case.

On the whole, therefore, a more stable arrangement is obtained than is the case when single repeaters are used, and this is particularly

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advantageous when more than one relay station is in circuit. A further marked advantage is that the double relay repeater is suitable for use in cord circuits at telephone exchanges for connection to various types of circuits.

The principal disadvantages are: (*a*) The apparatus is more complicated and expensive. (*b*) Communication from one end of the line to the other can only be completed by means of the relays, whereas in the single valve system the continuity of the circuit is maintained even if a valve fails. (*c*) The design of an artificial circuit for balancing purposes involves complex considerations and an approximation to a true balance has to suffice in commercial practice.

258. Artificial Circuits to represent Loaded Lines.—If loaded underground circuits are used then even if the circuit is sufficiently long to avoid end effects, difficulties still arise. This is because the so-called constants of the circuit are not quite uniform. Thus the

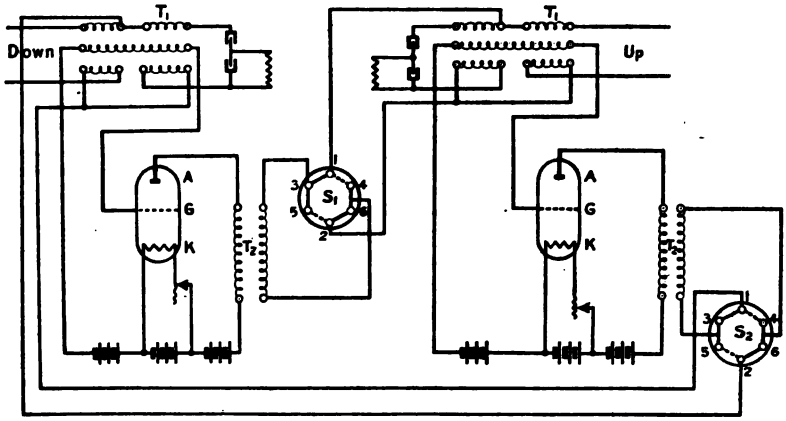

FIG. 182.—Double Relay Repeater. General Arrangement of Apparatus.

inductance of the loading coils may vary appreciably from the mean inductance, and the capacity in different lengths may also vary. The position of the last coil with respect to the end of the line has also a marked effect on the impedance, and the lumping of the inductance at fixed points contributes to the variations of impedance. If a circuit is subject to variations such as those pointed out, the impedance of an infinite line when measured with frequency between the limits of 300 and 1300 periods per second, commencing with the lower limit and making measurements at frequencies increasing by 40 periods per second, gives an undulatory curve as shown in Fig. 107. A method of constructing an artificial impedance to balance an infinite loaded line has been devised by Mr. C. Robinson, part author of the paper referred to. The problem, however, is much simplified if a capacity of 0·2 mf. is permanently bridged across the end of the loaded line, as this has the effect of partially smoothing out the impedance curve, and as the capacity only adds a very small transmission loss it is usually utilized. Mr. Robinson's method is as follows.

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It is known that the required impedance is one having a small angle in view of the fact that the line is loaded (see details of measurement of loaded lines in Chap. X.), and it is usually possible to make up the required artificial impedance by means of resistance and capacity only. From inspection of the measured results the limits of magnitude of the impedance are also known, and experience shows that the range of frequencies between 300 and 1300 cycles per second is sufficient. A number of standard curves are calculated within the required range of magnitude and frequency. These show the impedance-frequency characteristics of different combinations of resistance and capacity, the real and imaginary parts being plotted separately. The impedances so calculated consist of a simple resistance shunted by a capacity. The *measured* resistance and reactance components of the line impedance are drawn to the same scale on tracing paper. The resistance component is laid over the calculated resistance curves and it is usually possible to find one with the same slope. The actual value of the resistance can be obtained by adding resistance to the selected combination, since this does not affect either the resistance slope, or the magnitude of the reactance part. The reactance component is dealt with in a similar manner. When a curve with a suitable slope is found the reactance component can be adjusted to the required values by adding a series condenser. This, however, alters the slope of the reactance component, since the impedance of a condenser is inversely proportional to the frequency. It is, however, usually possible to find a capacity to be placed in series with the selected combination which gives an approximately correct reactance over the required range of frequencies. If, however, this is found to be impracticable, a different combination must be selected. The illustration taken is that of a lump-loaded line, but the method is applicable to any type of circuit.

It is also possible to arrive at the required impedance value by calculation, but the method involves a number of complex simultaneous equations, and in practice the method described gives satisfactory results and is preferred.

Equivalent impedances may also be calculated on the principles laid down in Chapters IV. and VI. Those methods, however, are particularly suitable when both attenuation and impedance are involved, whereas in the case investigated in the preceding paragraphs only impedance is in question. Moreover, the method of T and π circuits is strictly correct for one frequency only. If, however, the impedance network is made up of equivalent circuits which represent very short electrical lengths, the network will not vary appreciably from the line impedance required, but in such a case the network would be complex and might be unduly large.

259. The Van Kesteren System of Relay Working.—In order to overcome the difficulties in working associated with the balancing systems just described, Mr. A. S. J. Van Kesteren, a Dutch Telegraph Engineer, has proposed to utilize two trunk lines for each telephone conversation, one line being reserved for listening at each end. (British Patent Specification, No. 29165, 1913.) The relays are joined up for working in one direction only as shown on Fig. 183, the grid side of the relay being joined to the side of the circuit which is reserved at the distant end for speaking in each

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case, thus at one end of the circuit A is the speaking line and A_1 the receiving line, as indicated by the direction of the arrows. At the other end of the line B_1 is the speaking line and B the listening line. The two lines are joined together at the ends as shown, in order that the combined circuit may be extended to subscribers by means of one pair of lines as indicated. This device appears to involve some transmission loss. A transformer can also be used to effect the connection, and balancing devices added. The 4-wire system has the advantage that "howling" and other difficulties incident to imperfect balance are entirely eliminated, and the interaction between relays is prevented, provided that the relays in the two lines are not too near each other. It may be expected that the net results would be that a larger magnification could be utilized than in balanced systems, and the deterioration of articulation and want of stability which take place due to imperfectly balanced

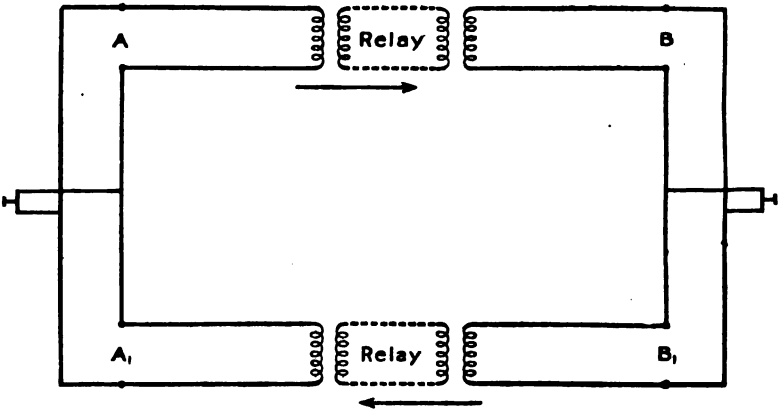

FIG. 183.—Van Kesteren Relay System. Explanatory Diagram.

repeater systems eliminated. How far these presumed advantages compensate for the necessity of using two pairs of wires is not at present known to the writer.

260. Signalling and Ringing Arrangements.—Signalling on relayed lines is usually necessary. The ideal method is to use a device which utilizes currents of sufficient frequency to actuate the thermionic relays. Signals could then be registered by means of electromagnetic relays with armatures tuned to vibrate at the selected ringing frequency. It is expected that such a method will shortly be introduced by the Post Office.

The ordinary ringing frequency usually adopted for ringing subscribers has a frequency of about 17 periods per second; the thermionic relay is not injured by such currents, but on the other hand it does not transmit such low frequency signals. Fig. 184 shows a method of ringing by means of generators operating at this frequency. The diagram is taken from a paper by Mr. A. B. Hart, read before the Institution of Post Office Electrical Engineers, on

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"Telephonic Repeaters," on Nov. 22nd, 1918. The paper is included in the series of professional papers published by the Institution.

261. Advantages of the Relay Method of Improving Transmission.—The improvement of telephonic transmission by means of telephonic relays gives promise of considerable development, and its close study will be advantageous to all who are interested in the study of transmission problems. This means of improving transmission has notable advantages over the attainment of the same means by improvements in subscribers' sending or receiving apparatus. In the first place, any method which would involve the scrapping of the enormous number of existing subscribers' instruments would be very costly. Not only is this so but a considerable increase in the efficiency of

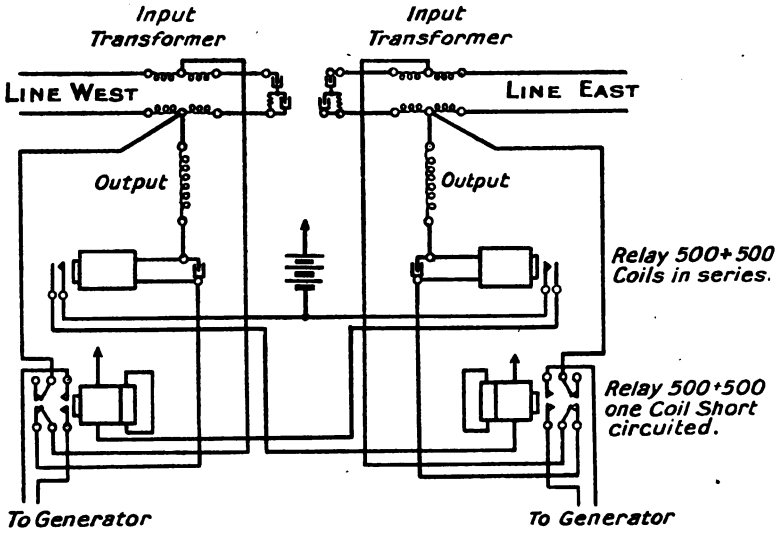

FIG. 184.—Arrangement for Ringing through Double Relay Repeaters

the transmitter and receiver is attended with difficulty. A considerable increase in the volume of speech at the sending end of a line would give rise to overhearing in some parts of the plant, which has been designed for existing conditions. Again, experience shows that any great increase in the volume of speech at the sending end of the line affects the ear painfully, and beyond a certain limit it is dangerous. Special steps have to be taken in such a case to cut out the receiver whilst a subscriber is speaking, and this is attended with considerable inconvenience, and gives rise to confusion. As regards the receiving apparatus, the telephone receiver is at present highly sensitive (see para. 183), and any apparatus of greater sensitivity increases the audibility of any cross-talk or other inductive disturbances.

It is here assumed that ordinary telephone frequencies are utilized for transmission, the conversion of the speech frequencies

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to those of a higher order, such as is involved in some modern proposals—for example, Squiers' wired wireless and similar proposals —are not referred to. In the case of the telephonic relay as described in this chapter, the volume of telephonic speech can be kept within normal limits whilst at the same time the possibility of extending the range of commercial telephone communication by this means is very great. It will also doubtless be used as a means of cheapening the cost of the plant necessary to obtain a fixed standard of transmission efficiency.

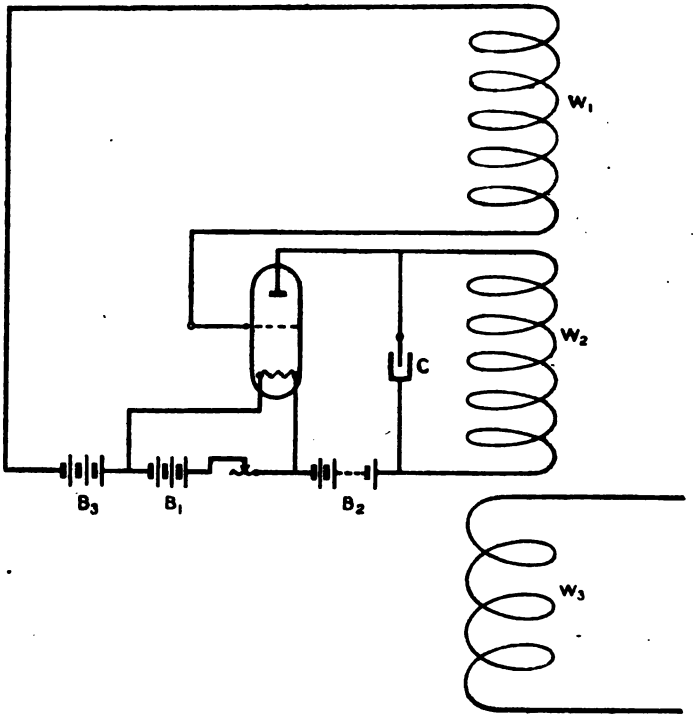

FIG. 185.—The Thermionic Valve arranged as an Oscillator.

262. The Thermionic Valve used as an Oscillator.—The use of this valve for the purpose named in the title is well known (see the "Handbook of Wireless Telegraphy and Telephony," by W. H. Eccles, p. 240, etc.). If the discharge circuit of a three-electrode thermionic valve is inductively coupled to the grid circuit through a transformer, a transient potential impressed on the grid circuit will be amplified in the discharge circuit; and a current will traverse the winding of the transformer in that circuit; as a result, a further induced potential will be impressed on the grid circuit via the transformer winding in that circuit, and this will again be magnified in the discharge circuit, and so on. The necessary arrangements for

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the production of electrical oscillations on this principle are shown in Fig. 185. A transformer with three windings W_1, W_2, and W_3 is provided, the windings W_1 and W_2 being connected as described, and the winding W_3 loosely coupled to the other part of the transformer. This winding is used to supply the generated alternating current as required. A condenser of suitable capacity is connected as shown, and by varying the amount of the capacity a range of frequencies can be provided which will depend on the magnitudes of M and C. When the oscillator is used to provide sine wave alternating current of telephonic frequency between $2\pi f = 3000$ and $2\pi f = 10{,}000$, suitable values of inductance are obtained by means of two air-core coils each having a resistance of 27 ohms, an inductance of 250 millihenrys, and a variable capacity of a total value of 2 mf. The best value of the mutual inductance M between the windings W_1 and W_2 and also the value of the capacity C is discussed in a "Note on the Production of Continuous Electrical Oscillations by the Three-electrode Valve," by E. V. L. Appleton, M.A., B.Sc., in the *Electrician* for December 27th, 1918. The winding W_3 may be wound to give some convenient value of inductance, but it should be loosely coupled. It may be mentioned that the battery in the grid circuit is sometimes dispensed with. An oscillator such as that described forms an extremely useful portable generator for telephonic sine-wave measurements, and it is a valuable adjunct to the apparatus described in Chapter X.

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APPENDIX I

Mathematical Proof of the Relations between the Voltage and Current in a Line of any Length with uniformly distributed Electrical Constants.

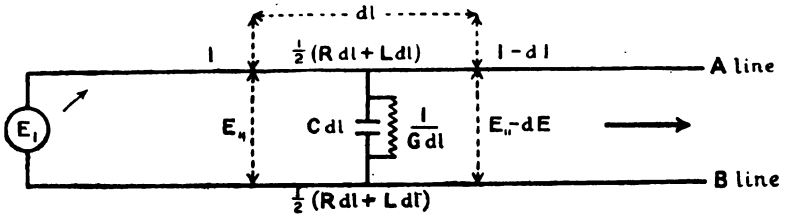

FIG. A.

Let R be the resistance of unit length of circuit.
,, G ,, leakance ,, ,, ,,
,, C ,, capacity ,, ,, ,,
,, L ,, inductance ,, ,, ,,
,, I ,, current at the length l from the sending end.
,, E ,, E.M.F. ,, ,, ,, ,,
,, $j = \sqrt{-1}$ indicating a lead of 90°.
,, γ be the propagation constant $= \beta + j\alpha$.

It is assumed that the voltage and current vary in a simple harmonic manner, their instantaneous values being—

$$e = E_{max} \sin \omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$i = I_{max} \sin (\omega t + \theta) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Then we have, for diminishing values of volts and current, the following equations:—

$$-\frac{de}{dl} = Ri + L\frac{di}{dt} . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$-\frac{di}{dl} = Ge + C\frac{de}{dt} . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Next, suppose the sine current to be represented by a vector denoting its maximum value I, then since the rate of change of a simple harmonic function is ω times the amplitude and leads by 90°, the vector quantity ωLI will be revolved forward through 90° as compared with the vector

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RI, and it may therefore be represented by $j\omega LI$, hence (3) and (4) may be converted into vector equations and written

$$-\frac{dE}{dl} = RI + j\omega LI = (R + j\omega L)I \quad . \quad . \quad . \quad (5)$$

$$-\frac{dI}{dl} = GE + j\omega CE = (G + j\omega C)E \quad . \quad . \quad . \quad (6)$$

Differentiating again with respect to l, we have

$$\frac{d^2E}{dl^2} = (R + j\omega L)(G + j\omega C)E \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{d^2I}{dl^2} = (G + j\omega C)(R + j\omega L)I \quad . \quad . \quad . \quad . \quad (8)$$

(7) and (8) may be written

$$\frac{d^2E}{dl^2} = \gamma^2 E, \text{ where } \gamma \text{ is a constant} \quad , \quad . \quad . \quad (9)$$

$$\frac{d^2I}{dl^2} = \gamma^2 I \quad ,, \quad ,, \quad ,, \quad . \quad . \quad . \quad (10)$$

Since the portion of (7) and (8) in brackets consists of constants, both (9) and (10) have the form of harmonic equations, and the solution of (9) is

$$E = Ae^{\gamma l} + Be^{-\gamma l} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is evident if (11) be differentiated, for we then obtain

$$\frac{d^2E}{dl^2} = \gamma^2(Ae^{\gamma l} + Be^{-\gamma l}) = \gamma^2 E$$

which agrees with (9).

$\frac{dI}{dl}$ is of the same form, and to obtain its value differentiate (11) with respect to l, we then have

$$\frac{dE}{dl} = \gamma(Ae^{\gamma l} - Be^{-\gamma l}) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substitute the value of $\frac{dE}{dl}$ in (12) from (5), we then obtain

$$I = \frac{\gamma(Ae^{\gamma l} - Be^{-\gamma l})}{-(R + j\omega L)} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

From (7) and (9) $\qquad \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad . \quad . \quad . \quad (14)$

Therefore (13) may be written

$$I = -\frac{(Ae^{\gamma l} - Be^{-\gamma l})}{\sqrt{\frac{R + j\omega L}{G + j\omega C}}} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Let $\sqrt{\frac{R + j\omega L}{G + j\omega C}}$ be written Z_0, then (15) becomes

$$I = -\frac{Ae^{\gamma l} - Be^{-\gamma l}}{Z_0} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

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To find the value of A and B. If $l = 0$, (11) and (16) may be written

From (11) $$E = Ae^0 + Be^0 = A + B$$

From (16) $$IZ_0 = -Ae^0 + Be^0 = -A + B$$

Whence $$\frac{E + IZ_0}{2} = B \quad \text{and} \quad \frac{E - IZ_0}{2} = A$$

Now since the values of E and I in A and B are the values obtained when $l = 0$, they are the current and voltage at the beginning of the line; call them E_1 and I_1, then (11) and (16) may be written

$$E = \frac{(E_1 + I_1Z_0)}{2} e^{-\gamma l} + \frac{(E_1 - I_1Z_0)}{2} e^{\gamma l} \quad . \quad . \quad . \quad (17)$$

$$I = \frac{\frac{E_1}{Z_0} + I_1}{2} e^{-\gamma l} - \left(\frac{\frac{E_1}{Z_0} I_1}{2}\right) e^{\gamma l} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If the line is indefinitely long $E_1 = I_1Z_0$ and $\frac{E_1}{Z_0} = I_1$, since the line is by supposition too long for any reflection effect, and (17) and (18) become

$$E = \frac{2E_1}{2} e^{-\gamma l} + \left(\frac{E_1 - E_1}{2}\right) e^{\gamma l} = E_1 e^{-\gamma l} \quad . \quad . \quad . \quad (19)$$

$$I = \frac{2I_1}{2} e^{-\gamma l} - \left(\frac{I_1 - I_1}{2}\right) e^{\gamma l} = I_1 e^{-\gamma l} = \frac{E_1}{Z_0} e^{-\gamma l}$$

Z_0 is the impedance of an indefinitely long or infinite line, and γ is the propagation constant of a decaying undulation; it is a vector by inspection of (14), and may be written $j\alpha + \beta = \gamma$, and we may consequently write

$$I = \frac{E_1}{Z_0} e^{-(j\alpha+\beta)l} = \frac{E_1}{Z_0} e^{-\beta l} e^{-j\alpha l} = \frac{E_1}{Z_0} e^{-\beta l} (\cos \alpha l - j \sin \alpha l) \quad . \quad (20)$$

In the following formulæ E and I, which are *general* values for the volts and current at any point on the line, will be discontinued, since the symbols E_1 and I_1 have been appropriated for the volts and current at the beginning of the line, and henceforth in this Appendix, E_2 and I_2, etc., will be used to denote the volts and current at any other point on the line.

(17) and (18) may be written, if $\gamma l = \theta$

$$E_2 = E_1 \frac{(e^{\gamma l} + e^{-\gamma l})}{2} - I_1Z_0 \frac{(e^{\gamma l} - e^{-\gamma l})}{2} = E_1 \cosh \theta - I_1Z_0 \sinh \theta \quad (21)$$

$$I_2 = I_1 \frac{(e^{\gamma l} + e^{-\gamma l})}{2} - \frac{E_1}{Z_0} \frac{(e^{\gamma l} - e^{-\gamma l})}{2} = I_1 \cosh \theta - \frac{E_1}{Z_0} \sinh \theta \quad . \quad (22)$$

Sending end impedance of a transmission line of any length open at the distant end.

In (22) $I_2 = 0$ at the point l, *i.e.* the receiving end, and therefore $I_1 \cosh \theta = \frac{E_1}{Z_0} \sinh \theta$, that is

$$Z = \frac{E_1}{I_1} = Z_0 \frac{\cosh \theta}{\sinh \theta} = Z_0 \coth \theta \quad . \quad . \quad . \quad (23)$$

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Sending end resistance of a transmission line closed at the distant end. In this case $E_2 = 0$ at the point l, and we get from (21)

$$E_1 \cosh \theta = I_1 Z_0 \sinh \theta$$

that is

$$Z_1 = \frac{E_1}{I_1} = Z_0 \frac{\sinh \theta}{\cosh \theta} = Z_0 \tanh \theta \quad . \quad (24)$$

To find the sending end current we have

$$\frac{E_1}{Z} = I_1 \text{ with the circuit open} \quad . \quad . \quad . \quad (25)$$

and

$$\frac{E_1}{Z_1} = I_1 \text{ with the circuit closed} \quad . \quad . \quad . \quad (26)$$

and by substitution of Z and Z_1 from (23) and (24) the values are obtained.

Current at the receiving end of a transmission line closed at the distant end.

By (22) $I_2 = I_1 \cosh \theta - \frac{E_1}{Z_0} \sinh \theta$.

Substitute the value of $I_1 = \frac{E_1}{Z_1} = \frac{E_1}{Z_0 \tanh \theta}$ from (24). We then obtain—

$$I_2 = \frac{E_1 \cosh \theta}{Z_0 \tanh \theta} - \frac{E_1}{Z_0} \sinh \theta$$

$$= \frac{E_1}{Z_0}\left(\frac{\cosh^2 \theta}{\sinh \theta} - \sinh \theta\right) = \frac{E_1 (\cosh^2 \theta - \sinh^2 \theta)}{Z_0 \sinh \theta}$$

$$= \frac{E_1}{Z_0 \sinh \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Voltage at the receiving end of a transmission line open at the receiving end.

Substitute the value of I_1 in (21), and we get

$$E_2 = E_1 \cosh \theta - \frac{E_1}{Z_0 \coth \theta} Z_0 \sinh \theta$$

that is

$$E_2 = \frac{E_1 \cosh^2 \theta}{\cosh \theta} - \frac{E_1 \sinh^2 \theta}{\cosh \theta} = \frac{E_1(\cosh^2 \theta - \sinh^2 \theta)}{\cosh \theta} = \frac{E_1}{\cosh \theta} \quad (28)$$

Impedance at the end of a transmission line closed at the receiving end.

The ratio of the impressed volts at the sending end to the received current is called the receiving end impedance.

We have $\frac{E_1}{I_2} = Z_2$ and by substitution of the value of I_2, Z_2 is found.

To show that the characteristic impedance $= \sqrt{ZZ_1}$.

From (23) and (24) we have

$$\sqrt{ZZ_1} = \sqrt{Z_0 \coth \theta \, Z_0 \tanh \theta} = Z_0 \quad . \quad . \quad (29)$$

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To show that the ratio $\frac{I_2}{I_1} = \frac{1}{\cosh \theta}$.

$$\frac{I_2}{I_1} = \frac{\frac{E_1}{Z_0 \sinh \theta}}{\frac{E_1}{Z_0 \tanh \theta}} = \frac{1}{\cosh \theta} \quad . \quad . \quad . \quad . \quad (30)$$

Hence $$\frac{I_1}{I_2} = \cosh \theta$$

Current at the receiving end of a transmission line closed at the receiving end by an impedance Za.

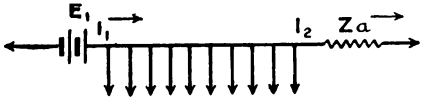

FIG. B.

The voltage across the receiving instrument is $I_2 Za = V_2$;

therefore by (21) $I_2 Za = V_2 = E_1 \cosh \theta - I_1 Z_0 \sinh \theta$

Also by (22) $$I_2 = I_1 \cosh \theta - \frac{E_1}{Z_0} \sinh \theta$$

From these equations I_2 may be found. Rearranging (21) and (22) we get

$$I_2 - \frac{E_1}{Za} \cosh \theta = -\frac{I_1}{Za} Z_0 \sinh \theta$$

$$I_2 + \frac{E_1}{Z_0} \sinh \theta = I_1 \cosh \theta$$

Hence $$\frac{I_2 - \frac{E_1}{Za} \cosh \theta}{I_2 + \frac{E_1}{Z_0} \sinh \theta} = \frac{-\frac{I_1}{Za} \times Z_0 \sinh \theta}{I_1 \cosh \theta} = -\frac{\frac{Z_0}{Za} \sinh \theta}{\cosh \theta}$$

that is $$I_2 \cosh \theta - \frac{E_1}{Za} \cosh^2 \theta = -I_2 \frac{Z_0}{Za} \sinh \theta - \frac{E_1}{Za} \sinh^2 \theta$$

$$I_2 \left(\cosh \theta + \frac{Z_0}{Za} \sinh \theta \right) = \frac{E_1}{Za} (\cosh^2 \theta - \sinh^2 \theta) = \frac{E_1}{Za}$$

that is $$I_2 = \frac{E_1}{Z_0 \sinh \theta + Za \cosh \theta} \quad . \quad . \quad . \quad . \quad (31)$$

Current at the sending end of a transmission line closed at the receiving end by a resistance Za.

In Fig. B the current I_1 is required.

(21) and (22) may be written

$$\frac{E_1 \cosh \theta}{Za} - I_1 \frac{Z_0}{Za} \sinh \theta = I_1 \cosh \theta - \frac{E_1}{Z_0} \sinh \theta$$

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that is $$E_1\left(\frac{\cosh\theta}{Za}+\frac{\sinh\theta}{Z_0}\right)=I_1\left(\cosh\theta+\frac{Z_0}{Za}\sinh\theta\right)$$

that is $$I_1=\frac{\left(\frac{\cosh\theta}{Za}+\frac{\sinh\theta}{Z_0}\right)E_1}{\cosh\theta+\frac{Z_0}{Za}\sinh\theta}$$

Multiply and divide by Za and we then have

$$I_1=\frac{E_1\left(\cosh\theta+\frac{Za}{Z_0}\sinh\theta\right)}{Za\cosh\theta+Z_0\sinh\theta}$$

that is $$I_1=\frac{E_1}{Z_0}\left(\frac{Z_0\cosh\theta+Za\sinh\theta}{Za\cosh\theta+Z_0\sinh\theta}\right) \quad . \quad . \quad . \quad (32)$$

From (21) and (22) the receiving end impedance and the sending end impedance are readily found. The receiving end impedance is defined as the ratio of the impressed volts at the sending end to the received current, or

$$\frac{E_1}{I_2}=Z_2=Z_0\sinh\theta+Za\cosh\theta \quad . \quad . \quad . \quad (33)$$

The sending end impedance is

$$\frac{E_1}{I_1}=Z_0\left(\frac{Za\cosh\theta+Z_0\sinh\theta}{Z_0\cosh\theta+Za\sinh\theta}\right) \quad . \quad . \quad . \quad (34)$$

The foregoing examples sufficiently illustrate the method. Further developments are given in Chapters III., VI. and XIII.

Development of the attenuation constant and the wave length constant.

The complex attenuation constant (14) is by inspection a vector quantity, and it may be written

$$\beta+j\alpha=\gamma=\sqrt{(R+j\omega L)(G+j\omega C)} \quad . \quad . \quad . \quad (35)$$

By squaring both sides we obtain

$$\beta^2+2j\alpha\beta+j^2\alpha^2=(R+j\omega L)(G+j\omega C)$$
$$=GR+j\omega LG+j\omega CR+(j\omega)^2LC$$

Equating the real and unreal parts we obtain

$$\beta^2-\alpha^2=GR-\omega^2LC \quad \text{and} \quad 2\alpha\beta=\omega LG+\omega CR$$

By squaring and adding we have

$$(\beta^2-\alpha^2)^2+(2\alpha\beta)^2=(GR-\omega^2LC)^2+(\omega LG+\omega CR)^2$$

that is

$$(\beta^2+\alpha^2)^2=G^2R^2-2\omega^2GRLC+\omega^4L^2C^2+\omega^2L^2G^2+2\omega^2LGCR+\omega^2C^2R^2$$
$$=G^2(R^2+\omega^2L^2)+\omega^2C^2(R^2+\omega^2L^2)=(G^2+\omega^2C^2)(R^2+\omega^2L^2)$$

or $$\beta^2+\alpha^2=\sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}$$

Also $$(\beta^2+\alpha^2)+(\beta^2-\alpha^2)=2\beta^2$$

$$\therefore\ 2\beta^2=\sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}+GR-\omega^2LC$$

and $$\beta=\sqrt{\tfrac{1}{2}\{\sqrt{(G^2+\omega^2C^2)(R^2+\omega^2L^2)}+(GR-\omega^2LC)\}} \quad (36)$$

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This is the real part of the complex attenuation constant, it is generally referred to as the attenuation constant.

Similarly $(\beta^2 + \alpha^2) - (\beta^2 - \alpha^2) = 2\alpha^2$

Whence $2\alpha^2 = \sqrt{(G^2 + \omega^2C^2)(R^2 + \omega^2L^2)} - (GR - \omega^2LC)$

or $$\alpha = \sqrt{\tfrac{1}{2}\{\sqrt{(G^2 + \omega^2C^2)(R^2 + \omega^2L^2)} - (GR - \omega^2LC)\}} \quad (37)$$

This is the wave length constant.

(35) implies not only that the current changes in magnitude by a given percentage (β) per unit length in an infinite line, but also that the sine wave revolves through a definite angle per unit length.

The imaginary part of γ (that is $j\alpha$) merely states that the wave has revolved α radians in unit length, *i.e.* the phase at unit distance differs from that at the beginning of the line by that angle. Consequently, at any length l it differs by αl radians. When $\alpha l = 2\pi$ then the wave has revolved through a complete cycle. The wave length is therefore

$$l = \frac{2\pi}{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If we know the frequency (*i.e.* the number of waves impressed on the line per second), then the velocity (V) of propagation is easily obtained, for the wave length is known and it is passed over in a time $t = \frac{1}{f}$ where f is the number of waves per second. We have $\frac{l}{t} = V$, and by substitution from (38)

$$\frac{2\pi}{\alpha t} = V$$

But $$t = \frac{1}{f} \text{ and therefore } V = \frac{2\pi f}{\alpha} \quad . \quad . \quad . \quad . \quad (39)$$

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APPENDIX II

Elementary Mathematical Proof of the Relations between the Voltage and Current in the Steady State in an Infinite Line with uniformly distributed Constants.

1. It is assumed that the current varies in a simple harmonic manner, its instantaneous value being $I_{max} \sin wt$. The symbols are the same as in Appendix I.

If dl is a very small fraction of unit length, the following will be the electrical values for that small length of the circuit :—

$R \times dl$ = Resistance of a very small fraction of unit length.
$G \times dl$ = Leakance ,, ,, ,, ,,
$L \times dl$ = Inductance ,, ,, ,, ,,
$C \times dl$ = Capacity ,, ,, ,, ,,

If E_1 is the voltage at the beginning of the length dl and E_2 the voltage at the end of this small length, then the voltage drop in the length dl is $E_1 - E_2$, and similarly in the next equal small length $E_2 - E_3$ where E_3 is the voltage at the end of the second small length dl.

$$\left.\begin{array}{l} \text{Let } E_1 - E_2 \text{ be called } dv \\ \text{,, } E_2 - E_3 \quad \text{,, } dv_1 \end{array}\right\} \text{etc.} \quad . \quad . \quad . \quad . \quad (1)$$

Although the voltage in such a line falls continuously along its whole length, we may suppose it to be unchanged in the length dl if that length be small enough. Call the current in this small length (say at the beginning of the line) I_1. The impedance of unit length is $R + j\omega L$, and therefore

$$\text{Impedance} \times dl = (R + j\omega L)dl \quad . \quad . \quad . \quad . \quad (2)$$

Now $\quad \text{Volts} = \text{Impedance} \times \text{Current} \quad . \quad . \quad . \quad . \quad (3)$

and therefore $\quad E_1 - E_2 = dv = (R + j\omega L)I_1 dl \quad . \quad . \quad . \quad . \quad (4)$

$= $ volts consumed in length dl

But current is also lost in the same length ; firstly, owing to direct leakage ; and secondly, because the dielectric gives rise to a capacity charge. The leakage in a small length dl is

$$\text{volts} \times \text{leakance} = E_1 G dl \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The current loss due to the condenser is

$$j\omega C E_1 dl = \text{loss due to capacity charge} \quad . \quad . \quad . \quad (6)$$

The combined loss due to (5) and (6) in a very small length is

$$I_1 - I_2 = di = E_1 G dl + j\omega C E_1 dl \quad . \quad . \quad . \quad . \quad (7)$$

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where I_1 is the current at the beginning of the small length and I_2 the current at the end of the length: this expression may be written

$$E_1(G + j\omega C)dl = di \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Divide (4) by (8) and we then get

$$\frac{dv}{di} = \frac{R + j\omega L}{G + j\omega C} \frac{I_1 dl}{E_1 dl} \text{ that is } \frac{dv}{di} \times \frac{E_1}{I_1} = \frac{R + j\omega L}{G + j\omega C} \quad . \quad (9)$$

2. This equation has the dimensions of an impedance multiplied by an impedance. If the necessary uniform conditions are fulfilled, as is the case in an infinite line, the impedance will be the same at any point on the line. This is evident from the following considerations:—

(*a*) The voltage and current are continually attenuated during their transmission along the line, owing to the combined capacity and leakance loss, and they must ultimately become so small after transmission over a sufficient length of circuit (an infinite circuit) that no electrical change is observed at the sending end even if the distant end be opened or closed.

(*b*) Whatever point on the line be taken the line in front is still supposed to be an infinite circuit, and therefore to be subject to an unvarying condition. It follows that

$$\frac{E_1}{I_1} = \frac{E_2}{I_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

since each is the impedance of the line at a different point and the impedance is everywhere equal, and therefore $\frac{E_1}{E_2} = \frac{I_1}{I_2}$. Subtract 1 from each side of this equation and we obtain

$$\frac{E_1}{E_2} - 1 = \frac{I_1}{I_2} - 1, \text{ that is } \frac{E_1 - E_2}{E_2} = \frac{I_1 - I_2}{I_2}, \text{ whence } \frac{E_1 - E_2}{I_1 - I_2} = \frac{E_2}{I_2} \quad (11)$$

but by (1) and (7)

$$\frac{E_1 - E_2}{I_1 - I_2} = \frac{dv}{di} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

therefore

$$\frac{dv}{di} = \frac{E_2}{I_2}$$

and therefore by (10)

$$\frac{dv}{di} = \frac{E_1}{I_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Substitute this value of dv in (9), and we get

$$\frac{E_1^2}{I_1^2} = \frac{R + j\omega L}{G + j\omega C}$$

whence

$$\frac{E_1}{I_1} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = Z_0 \quad . \quad . \quad . \quad (14)$$

Z_0 is called the characteristic impedance, it is the impedance of a very long line.

3. Again, from (4) and (8)

$$dvdi = (R + j\omega L)I_1 dl(G + j\omega C)E_1 dl$$

whence

$$\frac{dvdi}{E_1 I_1} = dl^2(R + j\omega L)(G + j\omega C)$$

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Since the impedance is constant at all points of the line this may be written from (3)

$$\frac{(di Z_0)di}{(I_1 Z_0)I_1} = \frac{di^2 Z_0}{I_1^2 Z_0} = dl^2(R + j\omega L)(G + j\omega C) \quad . \quad . \quad (15)$$

that is

$$\frac{di}{I_1} = dl\sqrt{(R + j\omega L)(G + j\omega C)} \quad . \quad . \quad . \quad . \quad (16)$$

That is to say, the ratio of the lost current di in any short length dl to the current I_1 at the beginning of that section is a constant fraction. This may also be expressed as $\frac{I_1 - I_2}{I_1} = \frac{di}{I_1}$, by dividing (7) by I_1. The current at the beginning of the second short length is I_2 and the current lost in that section is $I_2 - I_3$ if I_3 is the current at the end of the second short section.

Now $\frac{I_2 - I_3}{I_2} = \frac{I_1 - I_2}{I_1}$ for each equals $dl\sqrt{(R + j\omega L)(G + j\omega L)}$ from (16) *et seq.*

In a third short section we should have $\frac{I_3 - I_4}{I_3} = \frac{I_1 - I_2}{I_1}$, and so on to n short sections and since each of these ratios $= dl\sqrt{(R + j\omega L)(G + j\omega C)}$. Therefore $n\frac{I_1 - I_2}{I_1} = n \times dl\sqrt{(R + j\omega L)(G + j\omega C)}$, n being a number such that $n \times dl =$ unit length $= 1$.

Therefore $$n\left(\frac{I_1 - I_2}{I_1}\right) = n \times \frac{di}{I_1} = \sqrt{(R + j\omega L)(G + j\omega C)}$$

That is to say, in any unit length, whatever it may be (an inch, a mile, or a kilometre), the ratio of the current lost in that length, to the current at the beginning of it, is n times the fractional loss found for a very small length dl (where $n \times dl =$ unit length), and is measured by the quantity $\sqrt{(R + j\omega L)(G + j\omega C)}$, where R, L, G and C, are the constants per unit length. This quantity is known as the complex attenuation constant. It is complex because it involves phase, and is a vector. The constant is here denoted by γ, so that our expression becomes

$$n \times \frac{di}{I_1} = \gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad . \quad . \quad . \quad (17)$$

From inspection of the formula it may be seen that it consists of a real and an unreal part, and it may be written

$$\gamma = j\alpha + \beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad . \quad . \quad . \quad (18)$$

The two parts may be separated algebraically as shown in Appendix I.

4. It has been shown that the ratio of the lost current in any short length to the current at the beginning of that length, is a constant fraction anywhere in a uniform and infinite line, (16) *et seq.*, although the current is constantly altering in magnitude throughout the line. A little consideration will show that this is a case of the compound interest law, and that, consequently, the value of the current at any point is of the form $I_2 = e^{-\gamma l}$, where I_2 is the current and e is the base of the natural logarithms; also, γ is the complex attenuation constant per unit length, and l is the number of units of length.

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This may be proved as follows :—

Let the current at the beginning of the line be 1 and let γ be the fraction of current lost in the first mile. At the end of this mile the current will be $1-\gamma$, and this is the current at the beginning of the second mile. The same fraction of this beginning current, *i.e.* of $1-\gamma$, will be lost in the second mile as was lost in the first mile. So that at the end of the second mile the value of the remaining current will be $(1-\gamma)$ of $(1-\gamma) = (1-\gamma)^2$. Similarly, at the end of l miles it will be $I_2 = (1-\gamma)^l$. If instead of taking so large a length as a mile for the unit, an exceedingly small fraction of a mile be taken, such that instead of losing the fraction γ of the current per unit length, only the fraction $\frac{\gamma}{n}$ (where n is a very large number) is lost, then at the end of the first mile the current will have the value $\left(1-\frac{\gamma}{n}\right)^n$ and at the end of l miles, it will be

$$\left(1-\frac{\gamma}{n}\right)^{nl} = \text{I at point } l$$

It is known and can be verified, that if the algebraical quantity $\left(1+\frac{1}{x}\right)^x$ be expanded by the Binomial Theorem, and x then assumed to be a very large number, the Summation amounts to $2{\cdot}71828 = e =$ the base of the natural logarithms.

To make use of this fact, we convert $\left(1-\frac{\gamma}{n}\right)^{nl}$ to the form $\left(1+\frac{1}{x}\right)^x$. To effect this let $\frac{1}{x} = -\frac{\gamma}{n}$ then $n = -\gamma x$. Substitute this value of n in $\left(1-\frac{\gamma}{n}\right)^{nl}$ and we get $\left(1-\frac{\gamma}{n}\right)^{nl} = \left(1-\frac{\gamma}{-\gamma x}\right)^{-\gamma x l} = \left(1+\frac{1}{x}\right)^{x \times -\gamma l}$

$$= e^{-\gamma l} = e^{-(j\alpha+\beta)l} = I_2 \quad . \quad . \quad . \quad . \quad . \quad (19)$$

If the current instead of having the value 1 at the commencement is represented by A, then (19) becomes

$$I_2 = Ae^{-(j\alpha+\beta)l}$$

But the value of the current at the commencement of the line is $\frac{E_1}{Z_0}$ which may be substituted for A, and we finally obtain

$$I_2 = \frac{E_1}{Z_0} e^{-(j\alpha+\beta)l} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

5. This is the expression for a decaying undulation where β is the attenuation constant and α the wave length constant. The value of these constants is developed at the end of Appendix I.

It is a condition of this proof that the impedance is equal at any point of an infinite line, and subject to that condition the voltage and current in a transmission line with uniformly distributed constants (such as those stipulated in the proof) are attenuated in transmission along such a line in accordance with (20) ; or in other words, in accordance with the compound interest law. Now an infinite line in the

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literal sense does not exist, but a line with uniformly distributed constants and of any length fulfils the same compound interest law if the receiving end of the line is closed through a resistance having a value Z_0. This is evident if it be remembered that the line at any point has an impedance Z_0 measured from that point to the receiving end. If the line is cut at any such point and the receiving end impedance substituted by an artificial impedance Z_0, the impedance of the circuit at every point is unaltered and the attenuation of the volts and current from the sending end to the resistance Z_0 will evidently be unaltered.

6. If a direct current voltage is applied at the sending end of the circuit instead of an alternating current voltage, the formulæ derived are modified as follows, due to the fact that both ωL and ωC are dependent upon and proportional to the frequency factor, and therefore in the various direct current formulæ $\omega = 0$. Also by (37), App. I., if $\omega = 0$, then $\alpha = 0$, since this also depends upon the frequency. (14) therefore becomes

$$R_0 = \frac{E_1}{I_1} = \sqrt{\frac{R + 0}{G + 0}} = \sqrt{\frac{R}{G}} \quad . \quad . \quad . \quad (21)$$

(18) then takes the form

$$\gamma = \sqrt{(R + 0)(G + 0)} = \sqrt{RG} \quad . \quad . \quad . \quad (22)$$

(20) is also modified as follows:

$$I_2 = \frac{E_1}{R_0} e^{-(0+\beta)l} = \frac{E_1}{R_0} e^{-\gamma l} \quad . \quad . \quad . \quad . \quad (23)$$

It should be noted that in the direct current case β and γ are identical by (23), since $\alpha = 0$, but γ is retained for convenience. R_0 is substituted for Z_0 as the latter is an impedance, whereas R_0 is now the characteristic resistance.

Since $\frac{E_1}{R_0} = I_1$ the current at the sending end of the line, (23) may be written

$$I_2 = I_1 e^{-\gamma l}$$

whence

$$\frac{I_2}{I_1} = e^{-\gamma l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

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APPENDIX III

To find the Constants of an Artificial T Line which acts externally in the Steady State in the same way as a Line of any Length with uniformly Distributed Constants.

Direct Current Case.—Having given that the resistance, measured from the sending end of a line of any length, which is made up of uniformly distributed leakance and resistance, is constant when the line is closed by a resistance R_0, *i.e.* by its characteristic resistance (see Appendix II), it is required to find the values of the constants of an artificial circuit which shall satisfy this condition, and shall also fulfil the condition that the sent and received currents I_1 and I_2 satisfy the equation

$$\frac{I_2}{I_1} = e^{-\gamma l} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

as is the case with the uniform line in the conditions stipulated [App. II. No. (24)].

Let the artificial circuit be represented by the following diagram, R_1 and R_2 being independent resistances.

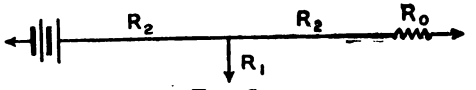

FIG. C.

It is required to find R_1 and R_2, R_0 being given, so that Fig. C always fulfils the foregoing conditions. By the elementary law of divided circuits the resistance (say Rx) of the equivalent circuit, Fig. C, measured from the sending end is

$$Rx = R_2 + \frac{R_1(R_2 + R_0)}{R_1 + R_2 + R_0} \quad . \quad . \quad . \quad . \quad (2)$$

By the conditions of the problem as stated above $Rx = R_0$. The sent current will divide at the junction of R_1 with the line R_2, inversely as the resistances R_1 and $R_2 + R_0$, and the proportion of the current flowing through R_0 to earth will be $\frac{R_1}{R_1 + R_2 + R_0}$. The received current I_2 is by similar reasoning

$$I_2 = \frac{I_1 R_1}{R_1 + R_2 + R_0} \text{ whence } \frac{I_2}{I_1} = \frac{R_1}{R_1 + R_2 + R_0}$$

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But by (1) $\frac{I_2}{I_1} = e^{-\gamma l}$. If the resistance at the sending end of the line is constant I_1 is constant, and by (1) and Appendix II. we have

$$\frac{I_2}{I_1} = e^{-\gamma l} = \frac{R_1}{R_1 + R_2 + R_0} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting R_0 for Rx in (2) we have

$$R_0 = R_2 + \frac{R_1(R_2 + R_0)}{R_1 + R_2 + R_0} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In (3) and (4) we have two unknown quantities R_1 and R_2 and their values may be found by means of two simultaneous quadratic equations, as follows, and the equations thus solved to satisfy paragraph 1. Clear (4) of fractions, and we get

$$R_2^2 + R_2R_0 + 2R_2R_1 + R_1R_0 = R_2R_0 + R_0^2 + R_1R_0$$

whence by cancelling like quantities we have

$$R_2^2 + 2R_2R_1 = R_0^2 \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If in equation (3) we represent $e^{-\gamma l}$ by A we may write (3) thus

$$AR_2 + AR_1 + AR_0 = R_1$$

whence $$R_0 = \frac{R_1}{A} - R_2 - R_1 = R_1\left(\frac{1}{A} - 1\right) - R_2 \quad . \quad . \quad (6)$$

Represent $\frac{1}{A} - 1$ by B, and then (6) may be written

$$BR_1 - R_2 = R_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Multiply (5) by B and (7) by $2R_2$, we then get

$$BR_2^2 + 2BR_2R_1 = BR_0^2 \quad . \quad . \quad . \quad . \quad (8)$$

$$2BR_2R_1 - 2R_2^2 = 2R_2R_0 \quad . \quad . \quad . \quad . \quad (9)$$

Subtract (9) from (8) we then have

$$BR_2^2 + 2R_2^2 = BR_0^2 - 2R_2R_0$$

that is $$(B + 2)R_2^2 + 2R_2R_0 - BR_0^2 = 0 \quad . \quad . \quad . \quad (10)$$

This equation (10) is of the form $aR_2^2 + bR_2 + c = 0$, the solution being

$$R_2 = \frac{-2R_0 \pm \sqrt{4R_0^2 + 4(B + 2)BR_0^2}}{2(B + 2)} \quad . \quad . \quad (11)$$

that is $$R_2 = \frac{-2R_0 \pm 2R_0(B + 1)}{2(B + 2)} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

that is $$R_2 = \frac{R_0\{(B + 1) - 1\}}{B + 2} = \frac{R_0B}{B + 2} \quad . \quad . \quad . \quad . \quad (13)$$

seeing that $2R_0(B + 1)$ is always positive.

But from (5) to (7) $B = \left(\frac{1}{A} - 1\right) = \frac{1}{e^{-\gamma l}} - 1 = e^{\gamma l} - 1$,

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Substitute this value in (13) and we then get

$$R_2 = R_0 \frac{(e^{\gamma l} - 1)}{(e^{\gamma l} + 1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Multiply and divide by $e^{-\frac{\gamma l}{2}}$ and we obtain

$$R_2 = R_0 \frac{\left(e^{\frac{\gamma l}{2}} - e^{-\frac{\gamma l}{2}}\right)}{\left(e^{\frac{\gamma l}{2}} + e^{-\frac{\gamma l}{2}}\right)}$$

that is

$$R_2 = \frac{R_0 \sinh \frac{\gamma l}{2}}{\cosh \frac{\gamma l}{2}} = R_0 \tanh \frac{\gamma l}{2} \quad . \quad . \quad . \quad . \quad (15)$$

If θ be written for γl then (15) becomes

$$R_2 = R_0 \tanh \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

To find R_1; formula (7) may be written

$$R_1 = \frac{R_0 + R_2}{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substitute the values of R_2 and B in (17), R_2 being written as in (14), and we get

$$R_1 = \frac{R_0 + \frac{R_0(e^\theta - 1)}{e^\theta + 1}}{e^\theta - 1} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

that is

$$R_1 = R_0 \frac{\{(e^\theta + 1) + (e^\theta - 1)\}}{e^\theta + 1} \times \frac{1}{e^\theta - 1}$$

that is

$$R_1 = \frac{R_0 2e^\theta}{e^{2\theta} - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Multiply and divide (19) by $e^{-\theta}$ we then have

$$R_1 = \frac{2R_0}{e^\theta - e^{-\theta}} = \frac{R_0}{\frac{e^\theta - e^{-\theta}}{2}}$$

that is

$$R_1 = \frac{R_0}{\sinh \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The artificial T circuit may therefore be represented as in Fig. D by substitution of the values of R_1 and R_2 in Fig. C and the elimination of the battery and the constant R_0.

This circuit fulfils the required conditions stated in paragraph 1, and the values found for R_1 and R_2 satisfy equations (3) and (4). The artificial circuit shown in Fig. D, therefore, is the equivalent T circuit of a uniform transmission line of any length, which has a resistance R_0 when closed by an impedance R_0, and it is the equivalent of a uniform line with an attenuation constant such that $\frac{I_2}{I_1} = e^{-\gamma l}$. If the artificial

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circuit is, as stipulated, always equal to the real circuit from its two ends, it will be equal when short-circuited or disconnected or when closed by resistances of any value, as well as when closed by R_0, and

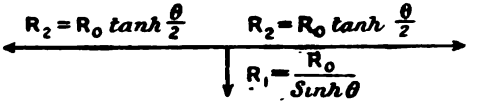

FIG. D.

such is in fact the case. The simple artificial circuit can therefore be operated upon as if it were the real circuit, and that too by means of Ohm's law, as shown in Chap. III.

Alternating Current Case.—In this case paragraph 1 holds good if the word "impedance" be substituted for the word "resistance." We then have two unknown impedances, which it will be convenient to call Z_5 and Z_6; these are vectors, and they replace the resistances R_1 and R_2 which are scalar quantities. The appropriate diagram is shown in Fig. E.

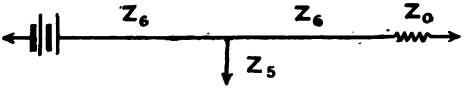

FIG. E.

Equations (3) and (4) are now modified as follows:

$$\frac{I_2}{I_1} = e^{-\gamma l} = \frac{Z_5}{Z_5 + Z_6 + Z_0} \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$Z_0 = Z_6 + \frac{Z_5(Z_6 + Z_0)}{Z_5 + Z_6 + Z_0} \quad . \quad . \quad . \quad . \quad (22)$$

Z_0 is now the characteristic impedance and all the quantities involved in the equations are vectors. These equations are exactly of the same form as equations (3) and (4), and in the general form in which they are expressed they must have similar roots to the direct current case; thus by simple substitution, instead of $R_2 = R_0 \tanh \frac{\theta}{2}$ we now have

$$Z_6 = Z_0 \tanh \frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and instead of $R_1 = \frac{R_0}{\sinh \theta}$ we now have

$$Z_5 = \frac{Z_0}{\sinh \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

It must, however, now be borne in mind that all the operations in the alternating current case are subject to the rules for the calculation of vectors; the more important of these rules are summarized in Chap. I. If the frequency of the A.C. is o we have the direct current case and equations (21) and (22) reduce to (3) and (4). (See Ex. 5. p. 102.)

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APPENDIX IV

To find the Constants of an Artificial π Line which acts externally in the Steady State in the same way as a Line of any Length having uniformly distributed Electrical Constants.

The problem is the same as that stated in Appendix III. except that a π line is dealt with in this case instead of a T line. Let the artificial circuit of the π line be represented by the following diagram :

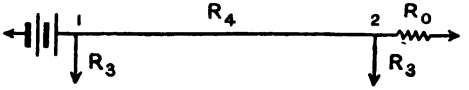

FIG. F.

The resistance of the above combination when measured from the junction 1 is as follows :—

Let the resistance of the combination from 1 in Fig. F, with R_3 disconnected at that point, be Rx, then the joint resistance of R_3 and Rx is

$$\frac{R_3 \times Rx}{R_3 + Rx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

By the law of divided circuits the resistance Rx may be written

$$Rx = R_4 + \frac{R_0 R_3}{R_0 + R_3} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substitute the value of (2) in (1) ; we then have

$$R_0 = \frac{R_3\left(R_4 + \frac{R_0 R_3}{R_0 + R_3}\right)}{R_3 + R_4 + \frac{R_0 R_3}{R_0 + R_3}} \quad . \quad . \quad . \quad . \quad (3)$$

Clearing of fractions we get

$$2R_0^2R_3 + R_3^2R_0 + R_4R_0^2 + R_0R_4R_3 = R_0R_4R_3 + R_3^2R_4 + R_0R_3^2 \quad (4)$$

whence

$$R_0^2 = \frac{R_4R_3^2}{R_4 + 2R_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$R_0 = R_3\sqrt{\frac{R_4}{R_4 + 2R_3}} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The current I_2 passing through R_0 to earth is

$$I_2 = \frac{R_3}{R_0 + R_3} \times \text{the current in } R_4 \quad . \quad . \quad . \quad (7)$$

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If Rx has the value stated in (2) the current I_1 will divide at 1 and the current in R_4 is

$$\frac{I_1R_3}{R_3+Rx} = \text{current in } R_4 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substitute this value in (7) and we have

$$I_2 = \frac{R_3}{R_0+R_3} \times \frac{I_1R_3}{R_3+Rx} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substitute the value of Rx in (9) and we have

$$I_2 = \frac{R_3}{R_0+R_3} \times \frac{I_1R_3}{R_3+R_4+\frac{R_0R_3}{R_0+R_3}} \quad . \quad . \quad (10)$$

that is $$\frac{I_2}{I_1} = e^{-\gamma l} = \frac{R_3^2}{R_0R_3+R_0R_4+R_3^2+R_4R_3+R_0R_3}$$

hence $$\frac{R_3^2}{e^{-\gamma l}} = 2R_0R_3 + R_0R_4 + R_3^2 + R_4R_3 \quad . \quad . \quad . \quad (11)$$

or $$R_0 = \frac{R_3^2(e^{\gamma l}-1) - R_4R_3}{R_4+2R_3}$$

If B be written for $e^{\gamma l} - 1$ we have

$$R_0 = \frac{BR_3^2 - R_4R_3}{R_4+2R_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Divide (5) by (12), we then have

$$\frac{R_0^2}{R_0} = R_0 = \frac{R_3^2R_4}{2R_3+R_4} \times \frac{2R_3+R_4}{R_3(BR_3-R_4)} = \frac{R_4R_3}{BR_3-R_4} \quad . \quad (13)$$

that is $$BR_0R_3 = R_4R_3 + R_4R_0$$

whence $$R_4 = \frac{BR_0R_3}{R_0+R_3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Insert this value of R_4 in (12) and we get

$$R_0 = \frac{R_3\left(BR_3 - \frac{BR_0R_3}{R_0+R_3}\right)}{\frac{BR_0R_3}{R_0+R_3}+2R_3}$$

Clearing of fractions we obtain

$$2R_0^2R_3 + 2R_0R_3^2 + R_0^2R_3B = BR_3^3$$

that is $$2R_0^2+2R_0R_3+BR_0^2=BR_3^2 \quad . \quad . \quad . \quad . \quad (15)$$

or $$BR_3^2 - 2R_0R_3 - R_0^2(B+2) = 0$$

This is of the form $ax^2 + bx + c = 0$, where the value of x is

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The root R_3 is therefore equal to

$$\frac{-(-2R_0) \pm \sqrt{4R_0^2 - \{-4R_0^2B(B+2)\}}}{2B} \quad . \quad . \quad (16)$$

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The quantity under the square root sign is $\{(2R_0(B+1)\}^2$, and therefore (16) may be written

$$R_3 = \frac{2R_0 \pm 2R_0(B+1)}{2B} \quad . \quad . \quad . \quad . \quad (17)$$

$B + 1$ is always positive and (17) may be written

$$R_3 = 2R_0\frac{(1+B+1)}{2B} = R_0\frac{(B+2)}{B} = \frac{R_0(e^{\gamma l}-1+2)}{e^{\gamma l}-1} \quad (18)$$

that is

$$R_3 = R_0\left(\frac{e^{\gamma l}+1}{e^{\gamma l}-1}\right) \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Multiply and divide by $e^{-\frac{\gamma l}{2}}$ and we then have

$$R_3 = R_0\frac{e^{\frac{\gamma l}{2}} + e^{-\frac{\gamma l}{2}}}{e^{\frac{\gamma l}{2}} - e^{-\frac{\gamma l}{2}}}$$

If θ be written for γl we obtain

$$R_3 = R_0 \coth\frac{\theta}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

To obtain R_4 substitute the value of R_3 in (14) and we obtain

$$R_4 = \frac{BR_0R_3}{R_0+R_3} = \frac{R_0^2(e^\theta-1)\left(\frac{e^\theta+1}{e^\theta-1}\right)}{R_0\left\{1+\left(\frac{e^\theta+1}{e^\theta-1}\right)\right\}} = \frac{R_0(e^\theta+1)}{\frac{e^\theta-1+e^\theta+1}{e^\theta-1}}$$

that is

$$R_4 = \frac{R_0(e^{2\theta}-1)}{2e^\theta}$$

Multiply and divide by $e^{-\theta}$ and we get

$$R_4 = \frac{R_0(e^\theta - e^{-\theta})}{2} = R_0 \sinh\theta \quad . \quad . \quad . \quad (21)$$

The equivalent circuit diagram is therefore as follows :

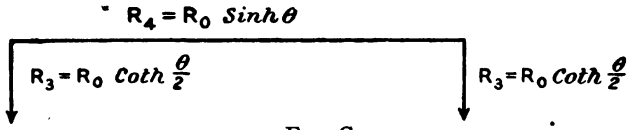

FIG. G.

The preceding proof refers to the direct current case, but it may be shown by similar reasoning to that at the end of Appendix III. that if A.C. impedances be taken instead of resistances, and Z_7, Z_8 and Z_0 be substituted for R_3, R_4 and R_0, a similar solution for the resulting equations would be found as those already found for R_3 and R_4 with the proviso that all the resulting calculations would require to be performed by vectors.

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APPENDIX V

To show that if $LG = CR$ then $\beta = \sqrt{RG}$, and that this is the Minimum Value of β.

In any telephone conductor with uniformly distributed constants we have by App. I. (36)

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + \tfrac{1}{2}(RG-\omega^2CL)} \quad . \quad . \quad . \quad (1)$$

$$= \sqrt{\tfrac{1}{2}\times\omega CR\left\{\sqrt{\left(1+\frac{\omega^2L^2}{R^2}\right)\left(1+\frac{G^2}{\omega^2C^2}\right)}+\frac{G}{\omega C}-\frac{\omega L}{R}\right\}} \quad . \quad . \quad (2)$$

$$= \sqrt{\tfrac{1}{2}\omega CR\left\{\sqrt{\left(1+\frac{\omega^2L^2}{R^2}+\frac{G^2}{\omega^2C^2}+\frac{L^2G^2}{C^2R^2}\right)}+\frac{G}{\omega C}-\frac{\omega L}{R}\right\}} \quad . \quad (3)$$

Now if $LG = CR$ then $\frac{L^2G^2}{C^2R^2} = 1$ and also $\frac{LG}{CR} = 1$

Substitute $\frac{LG}{CR}$ for 1 and for $\frac{L^2G^2}{C^2R^2}$ in (3). This then becomes

$$\beta = \sqrt{\tfrac{1}{2}\times\omega CR\left\{\sqrt{\left(\frac{\omega^2L^2}{R^2}+2\frac{LG}{CR}+\frac{G^2}{\omega^2C^2}\right)}+\frac{G}{\omega C}-\frac{\omega L}{R}\right\}} \quad . \quad (4)$$

The quantity $\frac{\omega^2L^2}{R^2}+2\frac{LG}{CR}+\frac{G^2}{\omega^2C^2} = \left(\frac{\omega L}{R}+\frac{G}{\omega C}\right)^2$ and therefore (3) becomes

$$\beta = \sqrt{\tfrac{1}{2}\times\omega CR\left(\frac{\omega L}{R}+\frac{G}{\omega C}+\frac{G}{\omega C}-\frac{\omega L}{R}\right)} = \sqrt{\tfrac{1}{2}\times\omega CR\times 2\frac{G}{\omega C}}$$

$$= \sqrt{RG} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, (3) is the equation for β in any and all cases whilst (4) gives $\beta = \sqrt{RG}$. The only difference between the two equations is that in (3), $1+\frac{L^2G^2}{C^2R^2}$ $\left\{\text{which may be written } \frac{LG}{CR}\left(\frac{CR}{LG}+\frac{LG}{CR}\right)\right\}$ is replaced by $2\frac{LG}{CR}$; the ratio of these two quantities is therefore

$$\frac{\frac{LG}{CR}\left(\frac{CR}{LG}+\frac{LG}{CR}\right)}{2\frac{LG}{CR}} = \frac{1}{2}\left(\frac{CR}{LG}+\frac{LG}{CR}\right) \quad . \quad . \quad . \quad (6)$$

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If this ratio is greater than 1 the numerator must be greater than the denominator $2\frac{LG}{CR}$, and therefore (3) must be greater than (4). If, however, LG = CR, (4) gives the minimum β. This is readily proved, for in $\frac{1}{2}\left(\frac{CR}{LG}+\frac{LG}{CR}\right)$ the expression in the brackets consists of a fraction and its reciprocal, and the minimum value of this is 2, by Fig. 2, Chap. I., and (91), Chap. I., and therefore $\frac{CR}{LG}=\frac{LG}{CR}$, whence CR=LG, *i.e.* the minimum value of (3) is found when CR = LG. In that case $\frac{1}{2}\left(\frac{CR}{LG}+\frac{LG}{CR}\right)=\frac{1}{2}(1+1)=1$, and therefore $\frac{LG}{CR}\left(\frac{CR}{LG}+\frac{LG}{CR}\right)=2\frac{LG}{CR}$. If CR does not equal LG, (6) must be greater than 1 and therefore (3) must be greater than (4) in all cases. The latter gives, therefore, the minimum value of β.

A simpler method of proving that $\beta_m=\sqrt{RG}$ might be given, but the method adopted permits of the subsequent simple proof that this is the *minimum* β.

To show that when $\beta=\sqrt{RG}$ then $\alpha=\omega\sqrt{CL}$.

As shown in Appendices I. and II.

$$\alpha=\sqrt{\tfrac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}-\tfrac{1}{2}(RG-\omega^2CL)}\,. \quad (7)$$

This may be written

$$\alpha=\sqrt{\left\{\tfrac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+\tfrac{1}{2}(RG-\omega^2CL)\right\}-(RG-\omega^2CL)} \quad (8)$$

The quantity in the large square brackets is the same as (1), and therefore

$$\alpha=\sqrt{\beta^2-(RG-\omega^2CL)} \quad . \quad . \quad . \quad . \quad (9)$$

Now when LG=CR, $\beta=\sqrt{RG}$ as shown in this Appendix. Therefore $\sqrt{RG}$ may be substituted for β in (8) in that case, and we have

$$\alpha=\sqrt{RG-RG+\omega^2CL}=\omega\sqrt{CL} \quad . \quad . \quad . \quad (10)$$

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APPENDIX VI

In any Uniform Transmission Line, if ωC is great in comparison with G and ωL is great in comparison with R the frequency factor disappears from the Attenuation Constant and the Circuit is said to be distortionless, both β and α are then simplified as shown in this Appendix.

$$\beta = \sqrt{\tfrac{1}{2}\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} + \tfrac{1}{2}(RG-\omega^2CL)} \quad . \quad (1)$$

See App. I. (36).

Extract the quantity under the inner root. We have

$$\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)} = \sqrt{\omega^2L^2\left(1+\frac{R^2}{\omega^2L^2}\right)\omega^2C^2\left(1+\frac{G^2}{\omega^2C^2}\right)}$$

$$= \omega^2CL\left(1+\frac{R^2}{\omega^2L^2}\right)^{\frac{1}{2}}\left(1+\frac{G^2}{\omega^2C^2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad (2)$$

Both of the latter quantities in the brackets are of the form $(1+x)^n$, and if R^2 is small in comparison with ω^2L^2, and G^2 is small in comparison with ω^2C^2 then, by a well-known algebraic approximation, $(1+x)^n = 1+nx$. In this case $n=\frac{1}{2}$ and $x=\frac{R^2}{\omega^2L^2}$. Also $x_1=\frac{G^2}{\omega^2C^2}$

We then have $(1+x)^{\frac{1}{2}}(1+x_1)^{\frac{1}{2}} = \left(1+\frac{x}{2}\right)\left(1+\frac{x_1}{2}\right)$ and similarly (2) may be written

$$\omega^2CL\left(1+\frac{R^2}{2\omega^2L^2}\right)\left(1+\frac{G^2}{2\omega^2C^2}\right) = \omega^2CL\left(1+\frac{R^2}{2\omega^2L^2}+\frac{G^2}{2\omega^2C^2}+\frac{R^2G^2}{4\omega^4C^2L^2}\right) \quad (3)$$

The last term is negligible and (3) then becomes

$$\left(\omega^2CL+\frac{R^2C}{2L}+\frac{G^2L}{2C}\right) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substitute this for the quantity under the inner root in (1) and we obtain

$$\beta = \sqrt{\tfrac{1}{2}\omega^2CL+\left(\frac{R^2C}{4L}+\frac{G^2L}{4C}+\tfrac{1}{2}RG\right)}-\tfrac{1}{2}\omega^2CL \quad . \quad (5)$$

Now the quantity in the inner brackets in (5) is a perfect square, and therefore (5) becomes

$$\beta = \sqrt{\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)^2}$$

that is

$$\beta = \frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

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To obtain the corresponding value of α the only change is that $\frac{1}{2}(RG - \omega^2CL)$ in the equation of β at the end of (1) becomes $-(\frac{1}{2}RG - \omega^2CL)$ in the equation for α as shown in (37), Appendix I. Equation (5) in this Appendix may then be changed to α as follows :—

If (4) be called A, then $\beta = \sqrt{\frac{1}{2}A + \frac{1}{2}(RG - \omega^2CL)}$ as in (5), and therefore α may be written $\alpha = \sqrt{\frac{1}{2}A - \frac{1}{2}(RG - \omega^2CL)}$ by substitution of A for the quantity under the inner root in (37), App. I., that is,

$$\alpha = \sqrt{\tfrac{1}{2}\omega^2CL + \left(\frac{R^2C}{4L} + \frac{G^2L}{4C} - \tfrac{1}{2}RG\right) + \tfrac{1}{2}\omega^2CL} \quad (7)$$

$$= \sqrt{\omega^2CL + \left(\frac{R}{2}\sqrt{\frac{C}{L}} - \frac{G}{2}\sqrt{\frac{L}{C}}\right)^2} \quad . \quad . \quad . \quad (8)$$

$$= \sqrt{\omega^2CL + \left(\frac{RC - GL}{2\sqrt{LC}}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If RC=GL the quantity in the brackets reduces to

$$\alpha = \omega\sqrt{CL} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Equation (6), however, is not limited to the condition that LG = CR, but of course to the conditions specified at the heading of this Appendix, and it should be ascertained by trial in any given case how far (9) differs from (10).

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APPENDIX VII

Proof of the Equation in accordance with which Fig. 90, Chap. IX. is Constructed and Examples of Application.

To show that $$\frac{\beta}{\beta m} = \frac{\frac{\sqrt{L}}{\sqrt{L_2}} + \frac{\sqrt{L_2}}{\sqrt{L}}}{2}$$

As stated in Chap. IX. in connection with Fig. 90, where

L = best inductance per unit length
L_2 = any inductance chosen for loading
L_1 = inductance of loading coil used

Represent $\left(\frac{R_1}{L_1} + \frac{G}{C}\right)$ by the letter A.

Then $AL = R$ by equation (4) Chap. IX.

$$\frac{\beta}{\beta m} = \frac{\frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L_2}{2}\sqrt{\frac{C}{L_2}}}{\frac{R + \left(\frac{R_1}{L_1} + \frac{G}{C}\right)L}{2}\sqrt{\frac{C}{L}}} \quad \ldots\ldots\ldots \quad (1)$$

$$= \frac{AL + AL_2}{AL + AL}\sqrt{\frac{L}{L_2}} = \frac{L + L_2}{2L}\sqrt{\frac{L}{L_2}}$$

$$= \frac{L\sqrt{L} + L_2\sqrt{L}}{2L\sqrt{L_2}} = \frac{L\sqrt{L}}{2L\sqrt{L_2}} + \frac{L_2\sqrt{L}}{2L\sqrt{L_2}}$$

$$= \frac{\frac{\sqrt{L}}{\sqrt{L_2}} + \frac{\sqrt{L_2}}{\sqrt{L}}}{2} = y \quad \ldots\ldots\ldots \quad (2)$$

If $\frac{L}{L_2}$ is called x then (2) becomes $\left(\frac{\sqrt{x} + \frac{1}{\sqrt{x}}}{2}\right) = y.$

If some other values βm_1 and y_1 are such that $\beta m_1\, y_1 = \beta$, where β is the same as in (6), Chap. IX., then, $\beta m y = \beta m_1\, y_1$ and by (51), Chap. VIII., this may be written

$$\sqrt{RC\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} \times y = \sqrt{R_1 C\left(\frac{R_1}{L_1} + \frac{G}{C}\right)} \times y_1$$

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If R and y are the only variables this reduces to

$$\sqrt{R} \times y = \sqrt{R_1} \times y_1$$

that is $$\sqrt{R}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \sqrt{R_1}\left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\right)$$

Squaring both sides we have

$$R\left(x + 2 + \frac{1}{x}\right) = R_1\left(x_1 + 2 + \frac{1}{x_1}\right) \quad . \quad . \quad . \quad (3)$$

Correction for Capacity and for $\frac{R_1}{L_1} + \frac{G}{C}$.—In connection with Fig. 90, Chap. IX., it is obvious from inspection of formula (51), Chap. VIII., that if nothing but the capacity alters, the values of β will vary as the square root of the capacity, and Fig. 89 is constructed on that basis.

The same remark applies to the value $\frac{R_1}{L_1} + \frac{G}{C}$, and Fig. H and a correcting table have been prepared for this (Table I., p. 389); previously to using this table, however, a prior correction must be made. From formula (4), Chap. IX., the value of L max. *i.e.* the best L, is inversely proportional to $\frac{R_1}{L_1} + \frac{G}{C}$. If this is called A, we have $\frac{L\,max.}{L_1 max.} = \frac{A_1}{A}$, where A is the value of $\frac{R_1}{L_1} + \frac{G}{C}$ corresponding to L max. and A_1 the value corresponding to L_1 max.

Now the x values in the horizontal line of Fig. 90 are the values $x = \frac{L\,max.}{L_2}$ where L_2 is any value of inductance it may be desired to use. Therefore for the *same* values of L_2, with different values of L max., we may write—

$$\frac{x}{x_1} = \frac{\frac{L\,max.}{L_2}}{\frac{L_1 max.}{L_2}} = \frac{L\,max.}{L_1 max.} = \frac{A_1}{A} \quad . \quad . \quad . \quad . \quad (4)$$

so that for a given value of L_2 the value of x varies inversely as $\frac{R_1}{L_1} + \frac{G}{C} = A$. Fig. H shows the ratio between the values of A and A_1, and therefore of the values of x and x_1. From the corrected value x_1, the corresponding value of y_1 can be found from Fig. 90.

Then if the correcting factor $\sqrt{\frac{\frac{R_1}{L_1} + \frac{G}{C}}{70}}$ from Table I. is called a_1, and βm is the value given in Fig. 90, the attenuation constant β for any circuit having a value of A differing from 70 is $\beta = \beta m \times a_1 \times y_1$.

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GRAPH SHOWING VARIATION OF VALUES OF x IN FIG. 90 WITH VARIATIONS OF $\frac{R_1}{L_1}+\frac{G}{C}$

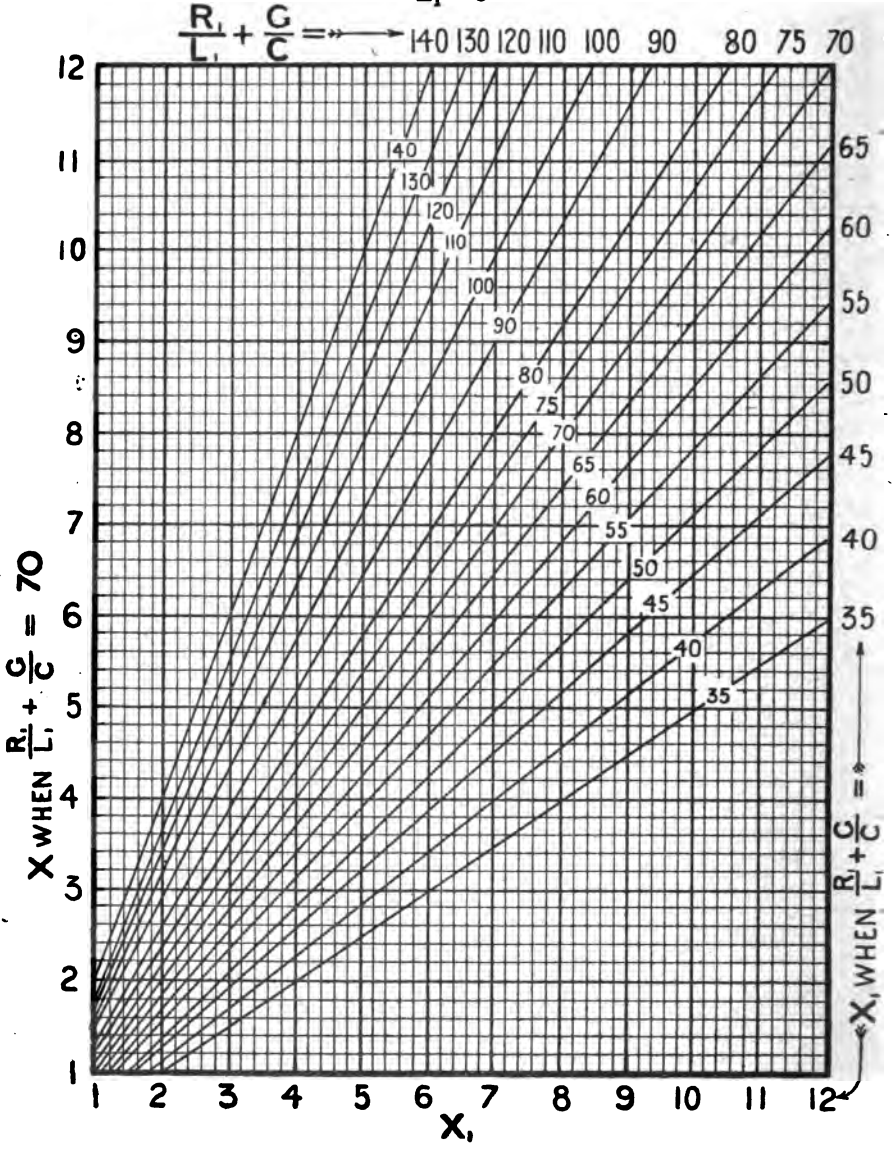

FIG. H.

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FIGURE 90, FURTHER EXAMPLES

Example 1.—It is proposed to use 55 m.h. per mile in a 100 lbs. copper circuit with a capacity of 0·065 mf. per mile. At the same time $\frac{R_1}{L_1} = 50$ and $\frac{G}{C} = 20$. The required "β" value may be read direct from Fig. 90, Chap. IX., thus: Find 55 m.h. in the horizontal line relating to 100 lbs. Cu at the foot of the graph, and then find the corresponding "β" value in the first vertical column on the right. It is 0·0116 nearly. It may be stated here that the β for the standard cable is 0·106, therefore the standard cable equivalent of the 100 lbs. circuit is $\frac{0·0116}{0·106} = 0·109$, and 9·14 miles of the 100 lbs. cable has the same attenuation as 1 mile of standard cable. (For standard cable, see Chap. XI.)

Example 2.—Assume that instead of 0·065 capacity the circuit in Example 1 has a capacity of 0·07 mf. per mile. Refer to Fig. 89, Chap. VIII. Find 0·07 in the horizontal line, and then in the vertical line find the corresponding value. Multiply the value found in Example 1 by this figure. We have $0·0116 \times 1·035 = 0·0120$, the required attenuation constant.

Example 3.—In a cable circuit with a capacity of 0·065 mf. per mile and $\left(\frac{R_1}{L_1} + \frac{G}{C}\right) = 70$, it is required to load a circuit to give a "β" of 0·016; what circuit and loading will give this result? Inspection of the graph shows three values, *i.e.* 100 lbs. with 24 m.h. per mile, 70 lbs. with 52 m.h. per mile, and 40 lbs. with 225 m.h. per mile. The choice is a matter of cost and convenience. The characteristic impedance must in this case be borne in mind, as the terminal loss would be different in the three cases.

When the value $\frac{R_1}{L_1} + \frac{G}{C}$ is more or less than 70, two corrections are necessary.

1st Correction to obtain the correct value of x in Fig. 90. *Example:* Suppose a 100 lbs. conductor is to be loaded with 84 m.h. per mile, the capacity being that provided in the graph, but with a loading coil having a resistance of 35 ohms per henry and $\frac{G}{C}$ being 20, we have then $\frac{R_1}{L_1} + \frac{G}{C} = 55$.

First refer to Fig. 90—find the value of the inductance to be used, *i.e.* 84 m.h., in the horizontal line relating to the 100 lbs. conductor.

Above this, on the top or x line under the curve, will be found the value 3. Next refer to Fig. H in this Appendix and find 3 in the vertical line at the left-hand side of the graph—then in a horizontal line with 3, find a point in the line marked 55, and vertically under this in the bottom horizontal line will be found 3·82. Now refer to 3·82 in the x or the top horizontal line of Fig. 90, the corresponding β in the 100 lbs. vertical column is 0·01095. If a large number of references for any value of $\frac{R_1}{L_1} + \frac{G}{C}$ are required, time is saved by permanently writing a new scale from Fig. H over the x line in Fig. 90.

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2nd Correction.—This value requires correction to account for the value 55 instead of 70. The table headed "Table of Multipliers" to be used in this connection has been prepared for the purpose. In the first column find the value of $\frac{R_1}{L_1}+\frac{G}{C}$ to be used in the calculation, *i.e.* 55, and in the other column the correcting factor will be found.

The required value of $\beta = \cdot01095 \times \cdot89 = \cdot00975$. This is very near the value found by formula (48), Chap. VIII.

If the "β" value in Fig. 90 is called β, and the capacity correcting multiplier is called C_1 and the $\frac{R_1}{L_1}+\frac{G}{C}$ multiplier (obtained as shown) is called a_1, then any β whatever is approximately given by $\beta \times C_1 \times a_1$.

For those expert in calculation, it may sometimes be quicker to calculate β from (51) or (48), Chap. VIII., if a number of references are required. In practice, however, with a slide rule, any required β can quickly be found by the graphs and tables.

If the circuits are superimposed, remember that a phantom circuit must be regarded as having double the weight of the conductor on which it is superimposed, and find the attenuation value for this double weight as before.

Owing to the fact that there are double the number of coils in superimposed circuits as compared with circuits not superimposed, the increase in resistance must be allowed for. The effect will be to increase the attenuation constant shown in the graph. With the type of loading and gauge of circuit superimposed up to the present an increase of 5 or 6 per cent. in the "β" found from the graph will give a fairly near approximation to the required "β." No fixed figure will, however, rigidly apply.

It should be noted that 100 lbs. Cu has a diameter of approximately 2 millimetres (it is 2·009 mm.), and that 20 lbs. Cu is approximately equivalent to No. 19 Brown and Sharpe gauge, which is 20·3 lbs. per mile—also that βm for different conductors varies inversely as the diameter of the gauge of conductor used. Bearing this in mind, other Continental and American attenuation values to a near degree of approximation may be deduced from the data given in Fig. 90 by the aid of Table No. 1, Chap. VIII., on the assumption that 2 mm. Cu is the same as 100 lbs. Cu and No. 19 Brown and Sharpe gauge the same as 20 lbs. Cu.

The error in βm in the former case is 0·3 per cent. and in the latter case 0·75 per cent., calculating on the basis of comparative weight of copper.

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TABLE NO. I.

Table of Multipliers to be used in connection with Fig. 90, Chap. IX., when $\left(\frac{R_1}{L_1}+\frac{G}{C}\right)$ is more or less than 70.

$\frac{R_1}{L_1}+\frac{G}{C}$	Multiplier $=\left(\sqrt{\frac{\frac{R_1}{L_1}+\frac{G}{C}}{70}}\right)$	
35	0·71	
40	0·76	
45	0·80	
50	0·85	
55	0·89	
60	0·93	
65	0·96	
70	1·00	
80	1·07	
90	1·13	
100	1·20	
110	1·25	
120	1·31	
130	1·36	
140	1·41	

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APPENDIX VIII

Relation between the Values of x and e^{-x}

Notes on Figs. J, K, and L.—These three curves show the relation between x and e^{-x} from $x = 0$ to $x = 13{\cdot}8$. The values of e^{-x} range from $e^{-x} = 1$ to $e^{-x} = 0{\cdot}000001$.

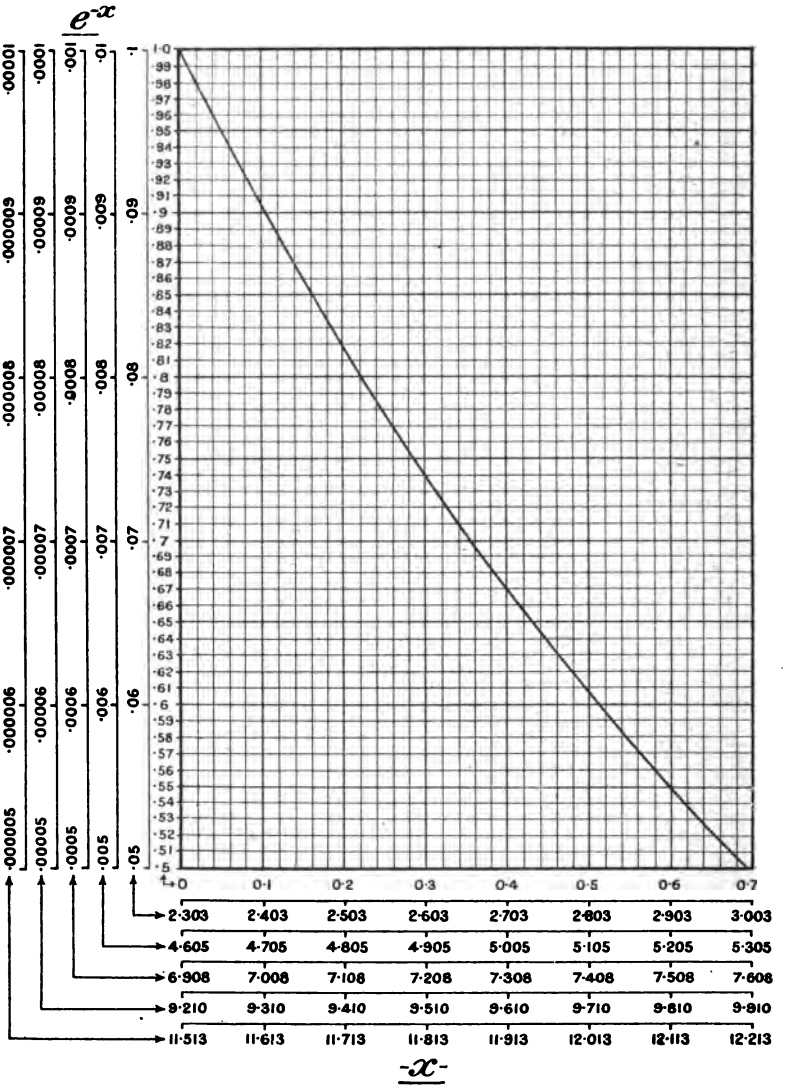

FIG. J.—Relation between x and e^{-x}.

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Advantage is taken of the fact that the curves relating e^{-x} and x continually repeat themselves after passing through a definite cycle of values of e^{-x}; thus in Fig. J all the figures (values of e^{-x}) in each vertical column vary uniformly from the figure at the top of each

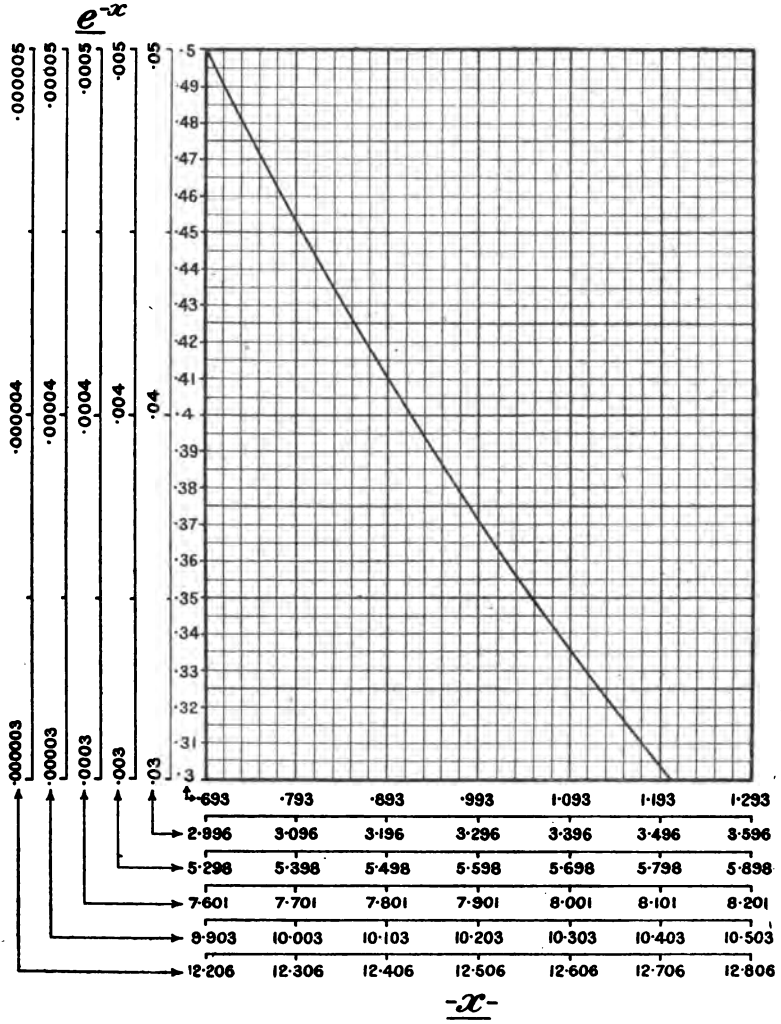

FIG. K.—Relation between x and e^{-x}.

column to half that value at the bottom of the column, and the natural logarithm x therefore varies in the way shown by the curve in each case. The actual numerical values of x are, however, of course different, as shown by the horizontal lines at the foot of the curve. Each vertical scale is connected with a definite horizontal scale, and the relative columns

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are indicated in all cases by arrows at the junctions of the two sets of lines. The same principle applies to all three curves. It will be observed that the figures cover a very wide range of values and that very small values of e^{-x} can be read to the same accuracy as relatively large ones.

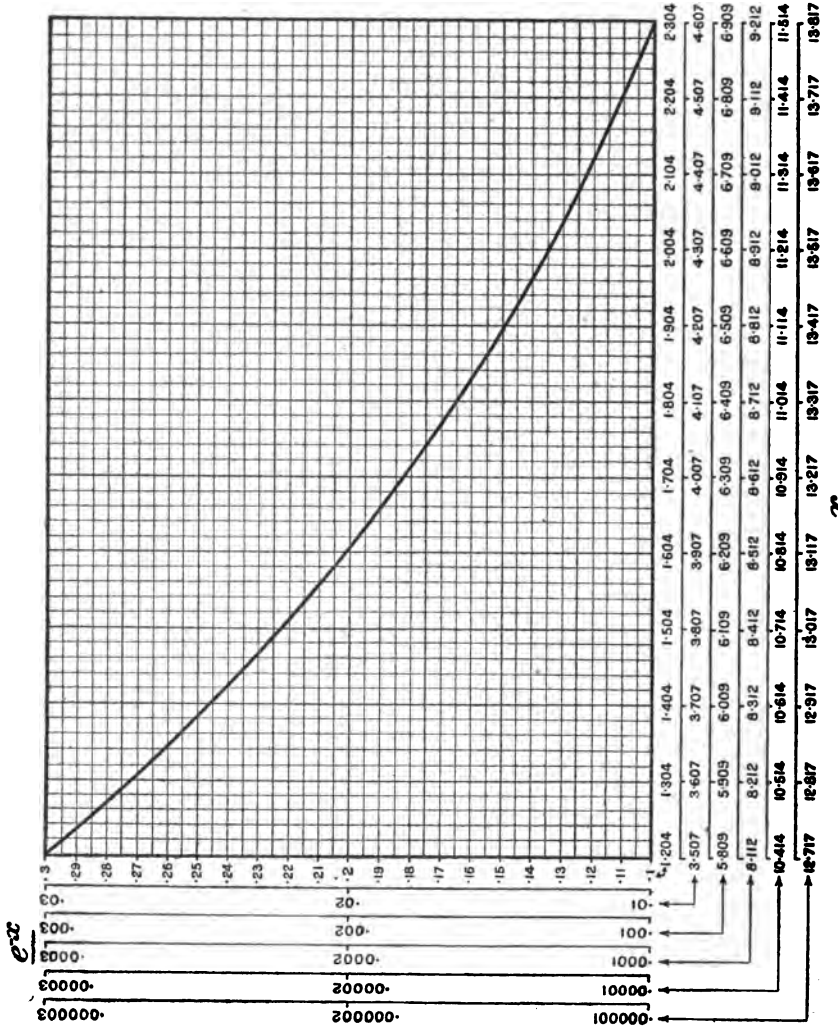

FIG. L.—Relation between x and e^{-x}.

Example 1.—Find the value of x when $e^{-x}=1$ and when $e^{-x}=0{\cdot}00001$.
Answer.—When $e^{-x}=1$, $x=0$, when $e^{-x}=0{\cdot}00001$ then $x=11{\cdot}513$.
Example 2.—The value of x is 5·498. What is the value of e^{-x}?
Answer.—See Fig. K for the value of x, in the third horizontal line from the top. The corresponding vertical value is 0·0041.
Example 3.—If $e^{-x}=0{\cdot}0025$ find x. See Fig. L.
Answer.—$x=5{\cdot}991$, nearly.

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